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Cubic differential systems with invariant straight lines of total multiplicity eight and four distinct infinite singularities

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ABSTRACT

In this article we prove a classification theorem (Main Theorem) of real planar cubic vector fields which possess four distinct infinite singularities and eight invariant straight lines, including the line at infinity and including their multiplicities. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of invariant polynomials. The algebraic invariants and comitants allow one to verify for any given real cubic system with four infinite distinct singularities whether or not it has invariant lines of total multiplicity eight, and to specify its configuration of lines endowed with their corresponding real singularities of this system. The calculations can be implemented on computer.

1. Introduction and the statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1}$$

where $P,\ Q$ are polynomials in $x,\ y$ with real coefficients, i.e. $P, Q \in \mathbb{R}[x, y]$. We shall say that systems (1) are *cubic* if $\max(\deg(P), \deg(Q)) = 3$.

Let

$$\mathbf{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

be the polynomial vector field corresponding to systems (1).

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A straight line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies

$$\mathbf{X}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$$

for some polynomial $R(x,y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients u, v, w of an invariant straight line belongs to $\mathbb{C} \setminus \mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real*. Note that, since systems (1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

To a line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ we associate its projective completion $F(X,Y,Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbf{P}_2(\mathbb{C})$, $(x,y) \mapsto [x:y:1]$. The line $Z = 0$ in $\mathbf{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. It follows from the work of Darboux (see, for instance, [5]) that each system of differential equations of the form (1) over $\mathbb{C}$ yields a differential equation on the complex projective plane $\mathbf{P}_2(\mathbb{C})$ which is the compactification of the differential equation $Qdx - Pdy = 0$ in $\mathbb{C}^2$. The line $Z = 0$ is an invariant manifold of this complex differential equation.

Definition 1.1. We say that an invariant affine straight line $f(x,y) = ux + vy + w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has multiplicity m if there exists a sequence of real cubic vector fields X_k converging to $\mathbf{X}$, such that each $\mathbf{X}_k$ has m (respectively $m-1$) distinct invariant affine straight lines $f_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0,0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m+1$ (respectively m).

We give here some references on polynomial differential systems possessing invariant straight lines. For quadratic systems see [6,18,19,21–25]; for cubic systems see [9,11,12,10,20,28,29]; for quartic systems see [27] and [32]; for some more general systems see [30,15–17].

According to [1] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [4].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities has been made in [10]. We also remark that a subclass of the family of cubic systems with invariant lines was discussed in [28] and [29]. More precisely, in these articles authors consider the cubic systems with exactly 7 invariant affine line considered with their “parallel” multiplicity (for more details of the confrontation of the results see the concluding comments at the end of this paper).

In this paper we classify the family of cubic systems with four distinct infinite singularities (real and/or complex), which possess eight invariant straight lines, including the line at infinity and taking into account their multiplicities (in the sense of Definition 1.1).

It is well known that for a cubic system (1) there exist at most 4 different slopes for invariant affine straight lines, for more information about the slopes of invariant straight lines for polynomial vector fields, see [2].

Definition 1.2. Consider a planar cubic system (1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

If a cubic system (1) possesses 8 distinct invariant straight lines (including the line at infinity) we say that these lines form a *configuration of type* $(3,3,1)$ if there exist two triplets of parallel lines and one additional line every set with different slopes. And we shall say that these lines form a *configuration of*