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# Bounded projections, duality and representations on large mixed norm spaces

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ABSTRACT

Recently, the boundedness of the projection on large Bergman spaces is obtained by Arroussi and Pau [2]. In this paper, we extend their results to the large mixed-norm Bergman spaces. In particular, using duality we obtain some representation theorems on such large mixed-norm spaces $A^{p,q}(\omega)$, $1 < p, q < \infty$.



## 1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $H(\mathbb{D})$ denote the class of all analytic functions on $\mathbb{D}$. Let $\omega$ be a radial weight function, that is, a positive function $\omega \in L^1(0,1)$. For $0 < p \le \infty$ and $0 < q < \infty$, the weighted mixed-norm Bergman spaces $A^{p,q}(\omega)$ consist of those functions $f$ in $H(\mathbb{D})$ such that

$$\|f\|^q_{p,q,\omega} := \int_0^1 M_p^q(r,f)\omega(r)dr < \infty.$$

Here, for $0 < p < \infty$, $M_p(r,f)$ are the integral means of $f$ defined by

$$M_p(r,f) = \left(\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi}\right)^{\frac{1}{p}}, \quad 0 \le r < 1,$$

and

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$$M_\infty(r,f) = \sup_{0\le\theta<2\pi} \left|f\left(re^{i\theta}\right)\right|, \quad 0 \le r < 1.$$

Moreover, for $0 < p < \infty$, the space $A^{p,\infty}(\omega)$ consists of those $f \in H(\mathbb{D})$ for which

$$\|f\|_{p,\infty,\omega} := \sup_{0\le r<1} M_p(r,f)\omega(r) < \infty.$$

When $\omega \equiv 1$, the spaces $A^{p,\infty}(\omega)$ is actually the classical Hardy space $H^p(\mathbb{D})$. In the case that $q = p$, $0 < p < \infty$, one obtains the weighted Bergman spaces $A^p(\omega) = A^{p,p}(\omega)$. For simplicity, $\|\cdot\|_{p,p,\omega}$ is denoted by $\|\cdot\|_{p,\omega}$ for $0 < p \le \infty$.

In this paper, we study the mixed-norm spaces with rapidly decreasing radial weights of the form $\omega = e^{-\varphi}$, where $\varphi \in \mathcal{C}^2(\mathbb{D})$ such that $\Delta\varphi(z) \ge C > 0$ for some constant $C$. If $\varphi(r) = -\alpha\log(1-r)$, $\alpha > -1$, then standard mixed-norm Bergman spaces are obtained. Throughout this paper, we consider weights which decrease faster than the standard weights including the weight

$$\omega(r) = (1-r)^\beta \exp\left(\frac{-\gamma}{(1-r)^\alpha}\right), \tag{1.1}$$

where $\alpha > 0$, $\gamma > 0$ and $\beta \in \mathbb{R}$. While an extensive study on standard Bergman spaces has been established, the theory of large Bergman spaces is not yet well understood, and some of the useful techniques valid for the standard Bergman spaces can fail to work in large Bergman spaces. However, there have been much efforts on characterizing those large Bergman spaces or operators acting on such a space in the disk [2,1,8,10,11]. Recently, in [2] Arroussi and Pau give the useful estimates for the reproducing kernel $K_z(\xi)$ of large Bergman space $A^2(\omega)$. As an application, they show that the Bergman projection

$$P_\omega f(z) = \int_{\mathbb{D}} f(\xi)\overline{K_z(\xi)}\omega(\xi)\,dA(\xi)$$

is bounded from $L^p(\omega^{p/2})$ to $A^p(\omega^{p/2})$ for $1 \le p < \infty$ and identify their dual spaces. Additionally, they suggest further problems on large Bergman spaces including the atomic decomposition theorem in the same paper. In the standard Bergman spaces case, the Möbius transformations play an important role for the atomic decomposition, but it can't be applicable in large Bergman spaces any more. For example, we can find some representations on standard Bergman spaces and Hardy spaces using the pseudohyperbolic metric balls in [6]. In this paper, we give a norm representation using the sequence $\{r_n\}$ given by [10] defined by the relation $\phi(r_n) = e^n$, $n \ge 0$ where $\phi$ denotes the positive increasing function defined by

$$\phi(r)^{-1} = \int_r^1 \omega(u)du. \tag{1.2}$$

We refer to page 11 for details. The main result of this paper is the obtention of norm representation theorems on large mixed-norm Bergman spaces $A^{p,q}(\omega)$ for a certain class $\mathcal{D}$ of weights defined in Section 2. Also, some basic properties of our weights as well as the associated distortion functions are presented in the same section. In Section 3, we extend the boundedness of the large Bergman projections to large mixed-norm Bergman spaces $A^{p,q}(\omega)$ for the case $1 \le p,q < \infty$, and the case $1 \le p < \infty$, $q = \infty$ (Theorem 3.2 and 3.3). Using the Bergman projection on large mixed-norm spaces, we give their dual spaces in Section 4 (Theorem 4.1). Finally, in the last section of this paper, we give a norm-representation theorem of $A^{p,q}(\omega)$ for $0 < p,q < \infty$ (Theorem 5.5) and the representation of the functions in $A^{p,q}(\omega)$, $1 < p,q < \infty$ as sums of kernel functions (Theorem 5.6).

*Constants.* In the rest of the paper we use the same letter $C$ to denote various positive constants which may change at each occurrence. Variables indicating the dependency of constants $C$ will be often specified.