J. Math. Anal. Appl. 423 (2015) 1183–1202

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global solvability for the porous medium equation with boundary flux governed by nonlinear memory

Jeffrey R. Anderson [a,*], Keng Deng [b]

[a] Department of Mathematical Sciences, Indiana University–Purdue University Fort Wayne, Fort Wayne, IN 46805, United States
[b] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504, United States

A R T I C L E   I N F O

Article history:
Received 29 May 2014
Available online 22 October 2014
Submitted by P. Sacks

Keywords:
Nonlinear diffusion equation
Global existence
Blow-up in finite time
Memory boundary condition

A B S T R A C T

We introduce the study of global existence and blow-up in finite time for nonlinear diffusion equations with flux at the boundary governed by memory. Via a simple transformation, the memory term arises out of a corresponding model introduced in previous studies of tumor-induced angiogenesis. The present study is also in the spirit of extending work on models of the heat equation with local, nonlocal, and delay nonlinearities present in the boundary flux. Specifically, we establish an identical set of necessary and sufficient conditions for blow-up in finite time as previously established in the case of local flux conditions at the boundary.

Published by Elsevier Inc.

1. Introduction

We initiate a study of global solvability for the nonlinear diffusion model

$$\begin{aligned} &u_t = \Delta\big(u^m\big) && \text{on } \Omega_T \\ &\nabla\big(u^m\big)\cdot\mathbf{n} = 0 && \text{on } (\partial\Omega\setminus\Sigma)_T \\ &\nabla\big(u^m\big)\cdot\mathbf{n} = u^q v && \text{on } \Sigma_T \\ &u = u_0 && \text{on } \overline{\Omega}\times\{0\} \end{aligned} \tag{1}$$

where

$$\begin{aligned} &v_t = u^p && \text{on } \Sigma_T \\ &v = 0 && \text{on } \overline{\Sigma}\times\{0\} \end{aligned} \tag{2}$$

* Corresponding author.
E-mail address: andersjr@ipfw.edu (J.R. Anderson).

http://dx.doi.org/10.1016/j.jmaa.2014.10.041
0022-247X/Published by Elsevier Inc.

Here, $m, p > 0$, $q \geq 0$, and $\Omega_T = \Omega \times (0, T)$, where Ω is a bounded domain in $\mathbf{R}^N$ having piecewise smooth boundary $\partial\Omega$ with outward pointing unit normal $\mathbf{n}$. Σ is a relatively open subset of $\partial\Omega$. The initial condition u_0 is a nonnegative, continuous function on $\overline{\Omega}$. We establish local existence, continuation, and subsolution comparison for weak solutions of (1)–(2). Our main result is that for $m > 1$, every solution of (1)–(2) is global if $0 < p + q \leq 1$, whereas if $p + q > 1$ and $u_0 > 0$, then all maximal solutions blow up in finite time. The analysis herein also applies in the case of the heat equation, $m = 1$. However, this has already been addressed in a previous paper [3].

Motivation for the study of (1)–(2) comes from both its identification as a subset of previously introduced models of tumor-induced angiogenesis and presenting a memory-type flux condition having had limited treatment in the literature. Formally integrating (2), one obtains the boundary condition

$$\nabla(u^m) \cdot \mathbf{n} = u^q \int_0^t u^p(\cdot, s)ds \quad \text{on } \Sigma_T$$

Our result may thus be viewed in analogy to the localized version of (2), namely $\nabla(u^m) \cdot \mathbf{n} = u^{q+p}$. As such, we extend results by Wolanski for the localized model [11].

As part of a model for the growth of new capillary networks ("angiogenesis"), initiated by a developing solid tumor, Levine et al. [6] introduced a general transmission condition, similar to (2), of the form

$$\begin{aligned} v_t &= f(x, t, u, v) + G(u)_t && \text{on } \Sigma_T \\ v &= v_0 && \text{on } \overline{\Sigma} \times \{0\}. \end{aligned}$$

In the application to angiogenesis, u and v are the concentrations of tumor-released growth factor outside and within the wall of a nearby capillary, respectively. The boundary condition arises in an effort to correctly represent uptake of growth factor within the capillary wall and its transport through the wall. Additionally, Σ represents the capillary wall. (See [6] for the entire system and a complete discussion of the model.)

The general program, beginning with more simplified systems such as (1)–(2), is to establish results on local and global solvability which are applicable to each of the various diffusion models contained within the full angiogenesis system introduced in [6]. Such results are of potential use in allowing future mathematical analyses of solutions for this fairly involved and complex system. Additionally, the local existence and subsolution comparison theory contained in the present work is an extension of known theories on diffusion models to include the case of a relatively new type of delay/memory boundary condition.

In our previous study of (1)–(2) for the case of the heat equation ($m = 1$) [3], a review of the literature is provided regarding memory conditions as incorporated in diffusion, reaction, and boundary flux in the case of uniformly parabolic models. Concerning models involving nonlinear diffusion and incorporating memory terms into the boundary flux, there appears to be only our previous work [2].

In the context of characterizing conditions under which these models are globally solvable, it is helpful to consider possible comparisons to corresponding localized models. As such, there is the aforementioned work by Wolanski [11], as well as the extension of this work to the reaction–diffusion model $u_t = \Delta(u^m) + u^\alpha$ with strictly positive initial data, u_0, and nonlinear flux $\nabla u \cdot \mathbf{n} = u^\beta$ at the boundary [8]. These authors have established that the model is globally solvable if and only if $\alpha \leq 1$, $\beta \leq \min\{1, (m+1)/2\}$, via suitable choices of subsolutions and supersolutions.

Although following from similar arguments applied to localized models, the local existence and comparison theories for (1)–(2) have not been previously addressed. In the next section, we state our results providing a general treatment of local existence for weak solutions along with subsolution comparison. Local solvability is established in Section 3, and subsolution comparison is proven in Section 4. The solution established herein is thus the maximal solution. However, consideration of uniqueness and supersolution comparison