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The sine-Gordon equation on time scales

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ABSTRACT

We formulate and discuss integrable analogue of the sine-Gordon equation on arbitrary time scales. This unification contains the sine-Gordon equation, discrete sine-Gordon equation and the Hirota equation (doubly discrete sine-Gordon equation) as special cases. We present the Lax pair, check compatibility conditions and construct the Darboux–Bäcklund transformation. Finally, we obtain a soliton solution on arbitrary time scale. The solution is expressed by the so-called Cayley exponential function.

1. Introduction

The sine-Gordon equation $\phi_{,xy} = \sin\phi$ is one of classical soliton equations. It has numerous applications in differential geometry [4,7,27,30], relativistic field theory [31], Josephson junctions [32,35], propagation of deformations along DNA double helix [38], displacements in crystals [17], domain walls in ferroelectric and ferromagnetic materials [24], mechanical transmission lines [32,34] and many others [2,15,25]. In spite of the long history the sine-Gordon equation and its numerous extensions and generalizations still attract attention of researchers [9,14,16,26,33,37]. In this paper we formulate and study an integrable extension of the sine-Gordon equation on arbitrary time scales, which includes the discrete case [29] and doubly discrete case (the Hirota equation) [21] as special cases.

A time scale $\mathbb{T}$ is any (non-empty) closed subset of $\mathbb{R}$ [19], including special cases like $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ and q-calculus. Time scales (or measure chains) were introduced in order to unify continuous and discrete calculus [20]. Below we recall several notions which will be used throughout the paper. We assume $t \in \mathbb{T}$. The forward jump operator is defined as

$$t^\sigma := \inf\{s \in \mathbb{T} : s > t\}.$$

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We also denote $\phi^\sigma(t) := \phi(t^\sigma)$. In the case $\mathbb{T} = h\mathbb{Z}$ the forward jump is a shift. Graininess is defined by

$$\mu(t) := t^\sigma - t.$$

In the case $\mathbb{T} = h\mathbb{Z}$ we have $\mu = h$ (time step). In the case $\mathbb{T} = \mathbb{R}$, we have, obviously, $\mu = 0$. The delta derivative is defined by

$$f^\Delta(t) := \lim_{s\to t} \frac{f(t^\sigma) - f(s)}{t^\sigma - s}.$$

Note that

$$f + \mu f^\Delta = f^\sigma. \tag{1.1}$$

The delta exponential function $e_a(t)$, see [6,19], satisfies the initial value problem

$$e_a^\Delta = a e_a, \qquad e_a(0) = 1, \tag{1.2}$$

where, in general, $a = a(t)$. We also have $e_a^\sigma = (1 + a\mu)e_a$.

In some applications (e.g., in trigonometry) another definition of the exponential function is much more convenient [12,13]. This is the Cayley exponential function which satisfies

$$E_a^\Delta = \frac{1}{2}a\big(E_a + E_a^\sigma\big), \qquad E_a(0) = 1, \qquad E_a^\sigma = \frac{1 + \frac{a\mu}{2}}{1 - \frac{a\mu}{2}} E_a. \tag{1.3}$$

In the continuous case ($\mathbb{T} = \mathbb{R}$) exponential functions become identical and for $a = \text{const}$ we have $e_a(t) = E_a(t) = e^{at}$.

Dynamic equations on time scales are counterparts of differential equations. They unify continuous and discrete dynamical systems [5]. Partial differential equations can be extended on time scales as well [1,22], including some soliton equations studied within the Hamiltonian framework [3,18,36]. In this case one needs to extend the notion of time scales on multidimensional spaces. In the next sections we consider the Cartesian product of two time scales $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$, where $T_1 \subset \mathbb{R}$ and $\mathbb{T}_2 \subset \mathbb{R}$. The elements of $\mathbb{T}_1$ and $\mathbb{T}_2$ are denoted by x and y, respectively.

This paper is a continuation of [10] where the Darboux–Bäcklund transformation for a class of 2×2 linear problems was constructed. We will construct this transformation for the integrable analogue of the sine-Gordon equation on time scales and compute a soliton solution expressed by the Cayley exponential function [13].

2. Lax pair

A standard Lax pair for the sine-Gordon equation $\phi_{,xy} = \sin\phi$ is given by:

$$\begin{aligned}
\Psi_{,x} &= \begin{pmatrix} i\zeta & -\frac{1}{2}\phi_{,x} \\ \frac{1}{2}\phi_{,x} & -i\zeta \end{pmatrix} \equiv i\zeta\sigma_3 - \frac{1}{2} i\phi_{,x}\sigma_2, \\
\Psi_{,y} &= \frac{1}{4i\zeta}\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \equiv \frac{1}{4i\zeta}(\cos\phi\sigma_3 + \sin\phi\sigma_1),
\end{aligned} \tag{2.1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices. We postulate the following time scale analogue for this Lax pair: