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Uncertainty principle for the Cantor dyadic group

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ABSTRACT

We introduce a notion of localization for functions defined on the Cantor group. Localization is characterized by the functional UC_d that is similar to the Heisenberg uncertainty constant for real-line functions. We are looking for dyadic analogs of quantitative uncertainty principles. To justify our definition we use some test functions including dyadic scaling and wavelet functions.

1. Introduction

Good time–frequency localization of function $f : \mathbb{R} \to \mathbb{C}$ means that both function f and its Fourier transform Ff have sufficiently fast decay at infinity. The functional called the Heisenberg uncertainty constant (UC) serves as a quantitative characteristic of this property. Smaller UCs correspond to more localized functions. The uncertainty principle (UP) expresses a fundamental property of nature and can be stated as follows. If $f \neq 0$ then it is impossible for f and Ff to be sharply concentrated simultaneously. In terms of the UC it means that there exists an absolute lower bound for the UC.

There are numerous analogs and extensions of this framework for different algebraic and topological structures. For example, the localization of periodic functions is measured by means of the Breitenberger UC [1]. For some particular cases of locally compact groups (namely euclidean motion groups, non-compact semisimple Lie groups, Heisenberg groups) a counterpart of the UC is suggested in [12]. The generalization of operator interpretation for the UC is discussed in [15]. These and many others related topics are described

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in the excellent survey [5]. As far as we know, the question of a quantitative UC for the Cantor dyadic group has not been referred in the literature. In this paper we try to understand what "good localization" means for functions defined on the Cantor dyadic group. So, a notion of the dyadic UC is suggested and justified. The existence of a lower bound is proved for the dyadic UC. We calculate this functional for dyadic scaling and wavelet functions and find well-localized dyadic wavelet frames. Some preliminary work is done in [9].

We do not discuss qualitative UPs in this paper. There exists a qualitative UP for a wide class of groups and the Cantor group belongs to the class (see p. 224, (7.1) [5]). It is easy to see that function $f_0 = \chi_{[0,1)} = \widehat{f}_0$, where $\widehat{f}$ is the Walsh–Fourier transform of f (see definitions and notations in Section 2), satisfies the extremal equality in this UP. There are a lot of results in this direction (see [8,7] and the references therein).

The paper is organized as follows. First, we introduce necessary notations and auxiliary results. In Section 3, we formulate the definition of the dyadic UC, discuss why the operator approach does not work here, prove the dyadic UP, answer the question how to calculate the dyadic UC in some particular important cases. In Section 4, we calculate the dyadic UC for Lang's wavelet and look for wavelet frames with small dyadic UCs.

2. Notations and auxiliary results

We use definitions and notations on the Cantor group and Walsh analysis from [6] and [13]. Let F be the set of sequences $\bar{x} = (x_k)_{k\in\mathbb{Z}}$, where $x_k \in \{0,1\}$ and either there exists $N(\bar{x}) \in \mathbb{Z}$ such that $x_{N(\bar{x})} = 1$ and $x_k = 0$ for $k < N(\bar{x})$ or $x_k = 0$ for all $k \in \mathbb{Z}$ (the zero element of the group F). The sum of $\bar{x} \in F$ and $\bar{y} \in F$ is defined by

$$x \,\overline{\oplus}\, y := \big(|x_k - y_k|\big)_{k\in\mathbb{Z}}.$$

Then $(F, \overline{\oplus})$ is an abelian group called `the Cantor dyadic group`.

Let $\lambda(\bar{x}) := \sum_{j\in\mathbb{Z}} x_j 2^{-j-1}$, then the map $\bar{x} \mapsto \lambda(\bar{x})$ is a one-to-one correspondence taking $F \setminus \mathbb{Q}_0$ onto $[0,\infty)$, where $\mathbb{Q}_0$ consists of all elements $(x_k)_{k\in\mathbb{Z}}$ such that $\lim_{k\to+\infty} x_k = 1$. Define the sum of numbers $\lambda(\bar{x})$ and $\lambda(\bar{y})$ by $\lambda(\bar{x}) \oplus \lambda(\bar{x}) = \lambda(\bar{x} \,\overline{\oplus}\, \bar{y}) = \sum_{j\in\mathbb{Z}} |x_j - y_j| 2^{-j-1}$. Denote the half-line $[0,\infty)$ equipped with operation $\oplus$ by $\mathbb{R}_+$. The set $\mathbb{R}_+$ is the standard interpretation of the group F, although the operation $\oplus$ is not associative on $\mathbb{R}_+$ (see for details [13, Sections 1.3, 9.1], [6, Sections 1.1, 1.2]). Since we do not need the associative property for our purpose, in the sequel, we use $\mathbb{R}_+$ instead of F. To simplify notations we denote $\lambda(\bar{x}) = x$. So, $x \oplus y = \sum_{j\in\mathbb{Z}} |x_j - y_j| 2^{-j-1}$ for $x, y \in \mathbb{R}_+$.

The set $\mathbb{R}_+$ is metrizable with the distance between $x, y \in \mathbb{R}_+$ defined to be $x \oplus y$. A function that is continuous from the $\oplus$-topology to the usual topology is called `W-continuous`.

`The Walsh-Fourier transform` of $f \in L_1(\mathbb{R}_+)$ is defined by

$$\widehat{f}(t) := \int_{\mathbb{R}_+} f(x) \mathrm{w}(t,x)\, dx,$$

where the function $\mathrm{w}(t,x) := (-1)^{\sum_{j\in\mathbb{Z}} t_j x_{-j-1}}$ is called `the generalized Walsh function`. It is an $\mathbb{R}_+$-analogue of a character of the group F. The Walsh–Fourier transform inherits many properties from the Fourier transform (see [13, Sections 9.2, 9.3]). For example, the Plancherel theorem holds

$$\int_{\mathbb{R}_+} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}_+} \widehat{f}(x)\overline{\widehat{g}(x)}\, dx,$$