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Variational approach to MEMS model with fringing field

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ABSTRACT

We study semilinear elliptic equations which are called MEMS model with fringing field. Using the variational method and dual approach, we show the multiple existence of positive solutions.

1. Introduction and main result

In this paper, we consider the following nonlinear elliptic equation:

$$\begin{cases} -\Delta u = \dfrac{\lambda(1+\delta|\nabla u|^2)}{(1-u)^p} & \text{in } \Omega, \\ 0 \le u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \ge 2$, $p > 1$, $\delta > 0$ and $\lambda > 0$ is a parameter. The purpose of this paper is to give the multiple existence result for solutions of (1.1) by using the variational method.

Eq. (1.1) is known as the steady state MEMS (Micro-Electro-Mechanical Systems) model. (See [16,19] for the derivation of MEMS.) Recently there have been many works on MEMS models. Here we briefly introduce the known results.

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Firstly when $\delta = 0$, (1.1) is reduced to the following problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda}{(1-u)^p} & \text{in } \Omega, \\ 0 \le u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Hereafter we call u a solution of (1.1), (1.2) if it belongs to the class $C^2(\Omega) \cap C^1(\bar{\Omega})$ and satisfies (1.1), (1.2) respectively. Then there exists $\lambda_0^* \in (0,\infty)$ (which is called a *pull-in threshold*) such that the following properties hold.

(i) If $0 < \lambda < \lambda_0^*$, (1.2) has a minimal stable solution u_λ. (For the definition of the stability, see Section 4.) When $\lambda = \lambda_0^*$, (1.2) has a unique solution $u_{\lambda_0^*}$. Finally if $\lambda > \lambda_0^*$, (1.2) has no solution. (See [7].)
(ii) When $\Omega \subset \mathbb{R}^N$ is a ball and $N \ge 8$, then we have $\lim_{\lambda \to \lambda_0^* - 0} \|u_\lambda\|_{L^\infty} = 1$. On the other hand if $\Omega \subset \mathbb{R}^N$ is a ball and $2 \le N \le 7$, or $\Omega \subset \mathbb{R}^2$ is a symmetric convex domain, then the solution branch has infinitely many turning points at some $\bar{\lambda}_0^* \in (0, \lambda_0^*)$. (See [6,11,12].)
(iii) When $\Omega \subset \mathbb{R}^N$ is strictly convex, then (1.2) has a unique solution for sufficiently small $\lambda > 0$. (See [5,20].)

Moreover there have been some results for the parabolic equation:

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda}{(1-u)^p} & \text{in } \Omega,\ t > 0, \\ u(x,0) = 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega,\ t > 0. \end{cases} \tag{1.3}$$

It is known that if $0 \le \lambda \le \lambda_0^*$, then (1.3) has a unique global solution $u(x,t)$, which converges to the minimal stable solution monotonically as $t \to \infty$. When $\lambda > \lambda_0^*$, quenching (touchdown) phenomena occur at a finite time $T^* < \infty$, that is, there exist $x_0 \in \bar{\Omega}$ and a sequence $t_n \nearrow T^*$ such that $\lim_{t_n \to T^*} u(x_0, t_n) = 1$. (See [8–10,15].)

On the other hand, Eq. (1.1) is less studied. The term $\delta|\nabla u|^2$ in (1.1) is called a *fringing field*, which was derived in [19] as a corner-corrected theory. Indeed, the correction term $\delta|\nabla u|^2$ is most effective where $\nabla u(x)$ becomes large, that is, near corners or edges.

In [17,19], it was observed by the numerical analysis that the fringing field drastically changes the structure of solutions. More precisely, the fringing field causes the pull-in threshold as in (1.2). On the other hand in [19], the authors showed numerically that if $\Omega \subset \mathbb{R}^2$ is a ball, then the infinite fold point structure does not appear for sufficiently small $\delta > 0$. A rigorous proof was obtained in [21]. The authors in [21] proved that (1.1) has at least two positive solutions by applying the bifurcation theory. Recently in [4], the authors showed the existence of solutions which approach to 1 as $\lambda \to 0$ at many points. These solutions are said to develop multiple point *ruptures*.

The aim of this paper is to analyze the change of the structure due to the fringing field by using the variational method. The main result of this paper is the following.

Theorem 1.1. *For any $\delta > 0$, there exists $\lambda_\delta^* \in (0,\infty)$ such that the following properties hold.*

(i) *If $\lambda > \lambda_\delta^*$, then (1.1) has no solution.*
(ii) *If $\lambda = \lambda_\delta^*$, then (1.1) has a unique positive solution.*