J. Math. Anal. Appl. 423 (2015) 1348–1369

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Index-2 elliptic partial differential-algebraic models for circuits and devices

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ARTICLE INFO

Article history:
Received 28 March 2014
Available online 30 October 2014
Submitted by R. Popovych

Keywords:
RLC networks
Semiconductors
Steady-state drift-diffusion
Coupled systems
Elliptic partial differential-algebraic equations (PDAEs)
Index

ABSTRACT

We consider an elliptic partial differential-algebraic model which arises in the modeling of an electric network that contains semiconductor devices. In this context, the electric network is described by linear differential-algebraic equations, while the semiconductor devices are described by nonlinear elliptic partial differential equations. The coupling takes place through the source term for the network equations and the boundary conditions for the device equations. Under the assumption that the fully coupled model has tractability index 2, we prove an existence result for this system. The extension of the notion of tractability index to a specific class of coupled systems is a further result of this paper.

1. Introduction

Often in circuit simulation, network designs are simulated on the basis of lumped network equations. These equations are usually derived from Modified Nodal Analysis (MNA). Generally, this modeling yields a system of differential-algebraic equations (DAEs). So-called index concepts [7] are used to classify these equations. The index roughly determines the number of inherent derivatives which are needed to derive the ordinary differential equation.

For the classical MNA equations, various index cases can be distinguished solely by structural means. That is, the network topology determines the index, see e.g. [5,6,13]. Using for instance the tractability index [9], one decomposes the set of variables accordingly and projects parts of the equations.

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http://dx.doi.org/10.1016/j.jmaa.2014.10.065
0022-247X/

Nowadays, downscaling as in semiconductor devices demands to include more and more former secondary effects in the electric circuit simulation. This leads directly to coupled systems of differential-algebraic equations (DAEs) for the electric network and partial differential equations (PDEs) for the semiconductor devices. The coupling has two parts. On the one hand, an additional source term occurs in the current balance of the electric network. On the other hand, the boundary conditions of the device equations depend on the time-dependent node potentials, which are genuine unknowns of the electric network.

In this paper we establish the existence of solution for an index-2 elliptic partial differential-algebraic equation. The fully coupled problem considered in our analysis has the following general structure (cf. [3]):

$$\begin{cases} E\dot{x} = Ax + \sigma + b(t), & t \in [t_0, t_1], \\ x(t_0) = x_0, \end{cases} \tag{1.1a}$$

$$\begin{cases} \mathcal{F}\big(r, u, \nabla u, \nabla^2 u\big) = 0, & r \in \Omega \subset \mathbb{R}^d, \\ \mathcal{B}(r, u, \partial u/\partial\nu, \eta) = 0, & r \in \partial\Omega, \end{cases} \tag{1.1b}$$

$$\sigma = Us(u), \tag{1.1c}$$

$$\eta = V^\top x \tag{1.1d}$$

for the unknown (x, σ, u, η).

(a) First, we have the initial value problem (1.1a), which consists of a differential-algebraic equation (DAE) in terms of the unknown $x : [t_0, t] \to \mathbb{R}^n$. Thereby, $b(t) \in \mathbb{R}^n$ describes a given input, while $\sigma \in \mathbb{R}^n$ expresses the coupling with the remaining part of the problem. We seek a solution in a corresponding functional space $\mathcal{W}_x$. This system represents the electric network equations.
(b) Secondly, we have the elliptic boundary value problem (1.1b) for the unknown u, which we seek in $\mathcal{W}_u$. The boundary condition $\mathcal{B} = 0$ depends on $\eta \in \mathbb{R}^k$, which will facilitate the coupling to the first subsystem. This system gives the relation for a distributed, static device (semiconductor).
(c) The PDE-to-DAE coupling is established by (1.1c). It maps the PDE unknown u, with the help of a given function $s : \mathcal{W}_u \to \mathbb{R}^\ell$, and a given matrix $U \in \mathbb{R}^{n\times\ell}$ to the DAE coupling σ. The term σ represents the device currents (through its terminals) padded with zeros.
(d) The DAE-to-PDE coupling is established by (1.1d). To this end, $V \in \mathbb{R}^{n\times k}$ is a given matrix. The term η represents the applied node potential.

For the extension of the index concept to these kinds of equations, we follow the approach introduced in [3]. The idea is to determine additional topological conditions on the coupling matrices which relate the DAE and PDE parts, such that the DAE index keeps its validity for the enlarged system. This system will be viewed as an electric network with a nonlinear controlled source (for the semiconductor device). Moreover, we will need to determine conditions under which the additional source terms have no further structural consequences.

These conditions are expressed in terms of a chain of projectors used to achieve a decoupled form of the DAE part of the system. It might be possible to adopt the quasi-Weierstrass approach proposed in [4], but this strategy is not pursued here.

For the coupled problem (1.1), we establish an existence result when the tractability index is 2, according to the generalization described above. This is the main result of this paper, since, to our knowledge, no existence results are available in literature for index-2 systems in the class of our study.

The work is organized as follows. Section 2 covers the modeling of the coupled system; both subsystems are described in detail and the coupling terms are defined. In Section 3 we recapitulate the tractability index concepts. We describe its application to the MNA equations, and we also give a topological interpretation of the index conditions. In Section 4 we state the main result, whose proof is presented in Section 5. First,