J. Math. Anal. Appl. 423 (2015) 1448–1468

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Large time behavior of solutions to the compressible Navier–Stokes equations with potential force

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ARTICLE INFO

*Article history:*
Received 22 July 2013
Available online 28 October 2014
Submitted by J. Guermond

*Keywords:*
Navier–Stokes equations
Potential force
Global existence
Time decay rates

ABSTRACT

The compressible Navier–Stokes equation with a potential external force is considered in $\mathbb{R}^3$ in the present paper. Under the smallness assumption on both the external force and the initial perturbation of the stationary solution in some Sobolev spaces, the existence theory of global solutions to the stationary profile is established. Furthermore, when the $\dot{H}^{-s}$ norm ($s \in (0, \frac{1}{2})$) of initial perturbation is finite, we obtain the optimal time decay rates of the solutions in $L^2$-norm. As a corollary, the $L^p$–$L^q$ $(3/2 < p \leq 2)$ type of the decay rates follows without requiring that the $L^p$ norm of initial perturbation is small.



## 1. Introduction

In this paper, we consider the initial value problem of the compressible Navier–Stokes equations with a potential force as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ u_t + (u \cdot \nabla)u + \dfrac{\nabla P(\rho)}{\rho} = \dfrac{\mu}{\rho}\Delta u + \dfrac{\mu + \lambda}{\rho}\nabla \operatorname{div} u - \nabla\phi(x), \\ (\rho, u)(0, x) = (\rho_0, u_0)(x) \to (\rho_\infty, 0), \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$. Here, $\rho = \rho(x,t) > 0$, $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $P = P(\rho)$ denote the density, velocity and the pressure function, respectively; $-\nabla\phi(x)$ is the time independent potential force; $\mu$, $\lambda$ are viscosity constants, satisfying $\mu > 0$, $2\mu + 3\lambda \geq 0$ which deduce $\mu + \lambda > 0$. In addition, $(\rho_\infty, 0)$ is the state of initial data at infinity, while $\rho_\infty$ is a positive constant and $P(\rho)$ is smooth in a neighborhood of $\rho_\infty$ with $P'(\rho_\infty) > 0$.

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http://dx.doi.org/10.1016/j.jmaa.2014.10.050

For the Navier–Stokes equations (1.1)$_1$–(1.1)$_2$ with potential force, the stationary solution $(\rho_*, u_*)(x)$ satisfies, cf. [17]

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P'(z)}{z}\mathrm{d}z + \phi(x) = 0, \qquad u_*(x) = 0. \tag{1.2}$$

There are many works which were devoted to proving the global existence, unique and time decay rates of solutions to the compressible Navier–Stokes equations with or without external forces, cf. [2,5–9,11,12,14–18,22,24] and references therein. In the following, we mainly mention some studies on the time decay rates of the solutions.

When omitting the external force, the stationary solution is just a constant. Matsumura and Nishida in [16] obtained the first global existence of small solutions when the initial perturbation is small in $H^3(\mathbb{R}^3)$. They also studied the $L^2$-norm decay rates in $H^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, see [15]. Moreover, the optimal $L^p$-norm time decay rates were proved by Ponce in [18]. Furthermore, the pointwise estimates of solutions were shown in [6,7,14] when the small initial perturbation in $H^N(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $N \geq [\frac{n}{2}] + 3$. Under the framework of $H^2(\mathbb{R}^3)$, by some elaborate estimates, the global existence of a strong solution and its optimal decay estimates were obtained in [24] when the initial data is bounded in $L^1$-norm. Recently, by using a nonnegative and negative Sobolev space $H^N(\mathbb{R}^3) \cap \dot{H}^{-s}(\mathbb{R}^3)$ ($s \in [0, 3/2)$) to replace $H^N(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $N \geq 3$, Guo and Wang in [5] developed a general energy method and obtained the following optimal time-decay estimates of solutions, i.e.

$$\left\|\nabla^k(\rho - \rho_*, u)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{k+s}{2}}, \quad \text{for } -s < k \leq N.$$

When a general external force is involved, the stationary solution $(\rho_*, u_*)$ may not be a constant. For this, when the initial disturbance belongs to $H^3(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$, the following convergence rate was obtained by Shibata and Tanaka in [20] for isentropic viscous fluid

$$\left\|\nabla(\rho - \rho_*, u - u_*)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{1}{2}+\kappa},$$

for any small constant $\kappa > 0$. The same decay rate for non-isentropic case was established by Qian and Yin in [19]. For the external potential force, based on the energy method and the spectral analysis on the linearized system (see (4.19)), Duan et al. studied the optimal time decay estimates

$$\left\|(\rho - \rho_*, u)(t)\right\|_{L^q(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})},$$

and

$$\left\|\nabla(\rho - \rho_*, u)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}},$$

when the initial perturbation belongs to $H^3(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, cf. [4] for $p = 1$ and [3] for $1 \leq p < 6/5$.

Motivated by [3,4], we prove $L^p$–$L^q$ type time decay estimates of solutions for $3/2 < p \leq 2$ by employing a negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$ to replace $L^p(\mathbb{R}^3)$. To be specific, we study the global existence and optimal time decay estimate of solutions to the problem (1.1) in both $H^2$-framework and $H^3$-framework.

First of all, when the initial perturbation is small in $H^2(\mathbb{R}^3)$, for the existence part, the difficulty mainly comes from the appearance of non-trivial stationary solutions. We should avoid the terms $\int_0^t \|\nabla^3(\rho - \rho_*)(\tau)\|_{L^2}\mathrm{d}\tau$ and $\int_0^t \|\nabla^4 u(\tau)\|_{L^2}\mathrm{d}\tau$ during the process of deducing *a priori* estimates. To overcome the difficulties, we employ a refined energy method. Moreover, when the initial perturbation is bounded in $\dot{H}^{-s}(\mathbb{R}^3)$, we obtain the optimal time decay estimates by the general energy method introduced in [5].