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A simple geometrical condition for the existence of periodic solutions of planar periodic systems. Applications to some biological models

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ABSTRACT

Using invariant regions for proving the existence of periodic solutions of periodic ordinary differential equations is a common tool. However, describing such a region is, in general, far from trivial. In this paper we provide sufficient conditions for the existence of an invariant region for certain planar systems. Our method locates the solution, in the sense that the region we determine evolves with time around the solution in the phase plane. Also, unlike other approaches, the construction does not depend on upper or lower bounds with respect to time of the functions involved in the system. The criterion is formulated for a general planar periodic ODEs system, and therefore it can be applied in very different contexts. In particular, we use the criterion to improve on previously known results on Holling's type II predator–prey periodic model, and on the classic periodic competition model.

1. Introduction

Non-autonomous ordinary differential systems are widely used for modeling natural phenomena. In particular, in that context, periodic functions are commonly involved and periodic solutions of the system are of interest. This is the case whenever one addresses problems in ecology, biology, epidemiology, geology, etc. In all these cases, environmental conditions that are often periodic, like light, temperature, humidity, rain, etc., highly influence the problem, and as a consequence periodic solutions are expected.

In this paper we present an original approach to the problem of detecting the existence of T-periodic solutions of planar periodic ordinary differential equation systems of the form:

$$\begin{cases} n' = f(t, n, p), \\ p' = g(t, n, p), \end{cases} \tag{1}$$

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where $'$ stands for the derivative d/dt, and f, $g \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^2)$ are periodic functions of t with the same period T. Our main result is a sufficient condition for the existence of at least one periodic solution of the system (1), which holds under certain hypotheses on the system (see the beginning of Section 2, but also Remark 2.3). Hence, we add a new approach to the variety of already available techniques for solving problems of this kind.

Our result has been motivated by the study of some models in population dynamics [25]; in fact, the result and the ideas behind are applicable, in concrete, to several predator–prey models, as well as to classical competition periodic models. Nevertheless, our ideas do not need to be formulated in a biological context.

A typical approach to prove the existence of periodic solutions is to determine an invariant region for the system (1) under study, where Brouwer's fixed point theorem can be applied to the induced Poincare map. See, for instance, Section 4.4 in [12] for an application to a two competing species periodic system. However, there is no general method for building such a region, and finding it is in fact a hard question. Typically (as in [12]) the construction of the invariant region is based on upper and lower bounds with respect to time of the functions f and g. In contrast, here we do not need or use these bounds (since we work locally).

A different approach is based on bifurcation techniques (e.g. [7]) and perturbation techniques (e.g. [26]). In [7] Cushing deals with a general n-dimensional periodic Kolmogorov system (see also the comments at the end of Section 2 herein) and derives the existence of a continuum of non-trivial solutions as a bifurcation phenomenon from a branch of trivial solutions. The underlying tool is a classical result due to Rabinowitz [28]. Compared to this approach, our main result here does not require the existence of a continuum of trivial solutions from where non-trivial ones bifurcate. Bifurcation appears again in [26], where the authors prove the existence of periodic solutions for small enough periodic perturbations by using techniques based on [16].

Also, another alternative is Mawhin's continuation theorem [13] (see, for instance, [11]). In this case, the existence result is stated in terms of the average value of the periodic parameters of the model. Compared to this, our method does not require any consideration on the average values of these parameters.

The paper is organized as follows. In Section 2 we prove our main result. We apply it in Section 3 to study a predator–prey model with Holling's II type functional response. To our knowledge, the results in this section are the first ones considering a periodic "closure term" [22,21]. In Section 4 we show how the ideas leading to the main result in Section 2 can be adapted to different settings, or even yield non-existence of positive periodic solution results. To this end, we revisit the standard competition periodic model [15] which still receives certain attention [1,23], and we provide new coexistence/species exclusion conditions. Finally, Section 5 contains some conclusions and observations about on-going research.

2. Main results

Our results are essentially of local nature and can be adapted to many different situations. However, in order to emphasize the main ideas and to make the reading as clear as possible, we will formulate some of the hypotheses required on the system (1) in a general fashion on the positive cone (see remarks after the main Theorem 2.1). One may observe that in fact our reasoning works the same in any appropriate region homeomorphic to the positive cone. The hypotheses that we require are:

- (H1) Every equation of the auxiliary system, obtained from (1) by setting $p' = 0$ and $n' = 0$, can be solved for p. That is, there exist unique functions φ and ψ such that $f(t, n, \varphi(t, n)) = 0$, $g(t, n, \psi(t, n)) = 0$.
- (H2) For each $t \in \mathbb{R}$ the curves $p = \varphi(t, n)$ and $p = \psi(t, n)$ meet just once in the positive cone. We denote the intersection point by $(\bar{n}(t), \bar{p}(t))$. Because of the periodicity on t of functions f and g (and, therefore, of φ and ψ) we have that $(\bar{n}(t), \bar{p}(t)) = (\bar{n}(t + T), \bar{p}(t + T))$, $\forall t \in \mathbb{R}$.
- (H3) As n increases and regardless the value of t, the function $\psi(t, n)$ is strictly increasing, while $\varphi(t, n)$ is strictly decreasing.