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Approximation properties for spaces of Bochner integrable functions

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ABSTRACT

For a finite measure space $(\Omega, \mathcal{A}, \mu)$, for a sub-$\sigma$-algebra $\mathcal{B} \subset \mathcal{A}$, and for a dual space X^*, having the Radon–Nikodým property, we show that every $\mathcal{A}$ measurable X^*-valued, Bochner integrable function has a best approximation in $L^1(\mathcal{B}, X^*)$. This extends a result of Papageorgiou, Shintani and Ando. For Banach spaces X, for which $L^1(\mathcal{A}, X)$ is an *L-embedded space*, we obtain a complete analogue of the main results of Shintani, Ando and Papageorgiou for increasing sequence of sub-σ-algebras.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let X be a Banach space. Let $L^1(\mathcal{A}, X)$ denote the space of X-valued Bochner integrable functions. It is an open problem if every $\mathcal{A}$ measurable X-valued, Bochner integrable function has a best approximation in $L^1(\mathcal{B}, X)$, for every sub-σ-field $\mathcal{B} \subset \mathcal{A}$. For a dual Banach space X^* having the Radon–Nikodým property (RNP), in this short note we give a simple proof of the result, that for any sub-σ-algebra $\mathcal{B} \subset \mathcal{A}$, $L^1(\mathcal{B}, X^*)$ is a proximinal subspace of $L^1(\mathcal{A}, X^*)$, i.e., for any $f \in L^1(\mathcal{A}, X^*)$, there is a $g \in L^1(\mathcal{B}, X^*)$ such that $d(f, L^1(\mathcal{B}, X^*)) = \|f - g\|$. Our proof is simple and different than the one given by N.S. Papageorgiou in [7] for a reflexive space X (as such spaces have the Radon–Nikodým property) and that of T. Shintani and T. Ando in [9], in the scalar-valued case. However their proofs have the advantage of obtaining the best approximation via minimizing sequences. We also show how the best approximation is related to characteristic functions.

We note that for a closed subspace $Y \subset X$, if $L^1(\mathcal{A}, Y)$ is a proximinal subspace of $L^1(\mathcal{A}, X)$, then $L^1(\mathcal{B}, Y)$ is a proximinal subspace of $L^1(\mathcal{B}, X)$. See [4] and [5] for general results on proximinality for spaces of Bochner integrable functions.

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We also recall in Section 3 the notion of simultaneous best approximation w.r.t. a monotonous norm and note the transitivity result that for a reflexive subspace $Y \subset X$, $L^1(\mathcal{B}, Y)$ is simultaneously proximinal in $L^1(\mathcal{A}, X)$, see [6] and [8].

For an increasing sequence of sub-σ-algebras, $\mathcal{B}_n$ increasing to $\mathcal{B}_\infty$, in general, we do not know if for an $f \in L^1(\mathcal{A}, X^*)$, a best approximation w.r.t. $L^1(\mathcal{B}_\infty, X^*)$ can be obtained as a weak limit of the best approximations from $L^1(\mathcal{B}_n, X^*)$? For a Banach space X for which $L^1(\mathcal{A}, \mu, X)$, under the canonical embedding, is the range of an L-projection (the so called *L-embedded spaces*, see [3, Chapter IV]), we show that for an $f \in L^1(\mathcal{A}, X^*)$, a best approximation w.r.t. $L^1(\mathcal{B}_\infty, X^*)$ can be obtained as a weak limit of the best approximations from $L^1(\mathcal{B}_n, X^*)$. This property of $L^1(\mathcal{A}, \mu, X)$ is well known when X is the scalar field. Thus we have a complete analogue of Theorem 7 of [9].

2. Proximinality results

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Since for a purely discrete measure μ, measurability w.r.t. a sub-σ-algebra makes no difference, we may and do assume that μ is a purely non-atomic measure. Also as functions are identified a.e., we may assume that $\mathcal{A}$ is complete w.r.t. μ. Since Bochner-integrable functions have a.e. separable range and as proximinality involves only approximation by sequences, we may assume that $\mathcal{A}$ is the completion of a countably generated algebra (see also the proof of Corollary 3.5 in [5]). Thus via the Borel isomorphism theorem (see [12, Section 3.3]), we may assume that $\Omega = [0, 1]$ and μ is the Lebesgue measure on the Lebesgue σ-field. Let $C([0, 1], X)$ denote the space of X-valued continuous functions equipped with the supremum norm. It is easy to see, via Singer's theorem (see [10] and [11]), that $C([0, 1], X)^*$ is identified with the space of X^*-valued countably additive regular Borel measures, equipped with the total variation norm. Now the canonical embedding $f \mapsto f d\mu$ of $L^1([0, 1], \mathcal{A}, X^*)$ in $C([0, 1], X)^*$ is an into isometry. We refer to the monograph [2] for all standard terminology from the theory of vector measures and Bochner integrable functions. For a Banach space Y embedded canonically in Y^{**}, by $Q : Y^{***} \to Y^*$ we always denote the canonical projection $Q(\Lambda) = \Lambda|Y$. We also note that for a closed subspace $Z \subset Y$, Z^{**} is identified canonically as the weak*-closed (and hence proximinal) subspace $Z^{\perp\perp}$ of Y^{**}. Also $Z^{\perp\perp} \cap Y = Z$.

Theorem 1. *Let X^* be a dual Banach space with RNP. Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive measure space. Let $\mathcal{B} \subset \mathcal{A}$ be a sub-σ-algebra. Then $L^1(\mathcal{B}, X^*)$ is a proximinal subspace of $L^1(\mathcal{A}, X^*)$.*

Proof. In view of the measure theoretic reduction discussed above we may assume that $\Omega = [0, 1]$, $\mathcal{A}$ is the Lebesgue σ-field and μ is the Lebesgue measure.

We recall from Chapter V of [2] that the conditional expectation is a linear projection $E : L^1(\mathcal{A}, X^*) \to L^1(\mathcal{B}, X^*)$, defined by $E(f) = g$, where $\int_B f\, d\mu = \int_B g\, d\mu$ for $f \in L^1(\mathcal{A}, X^*)$, $g \in L^1(\mathcal{B}, X^*)$, $B \in \mathcal{B}$. This is a projection of norm one.

Since X^* has the RNP, using the Lebesgue decomposition theorem (see [2, I.5.9]), it follows that $P : C([0, 1], X)^* \to L^1(\mathcal{A}, X^*)$ defined by $P(F) = dF_a$, where dF_a is the Radon–Nikodým derivative of the absolutely continuous part F_a, w.r.t. μ, is again a linear projection. As we have the total variation norm on the domain, it is easy to see that this projection is of norm one. This projection is an extension of E.

Thus by composing with Q, for the inclusion $L^1(\mathcal{B}, X^*) \subset L^1(\mathcal{A}, X^*)$, we have two linear projections of norm one, $P' : L^1(\mathcal{A}, X^*)^{**} \to L^1(\mathcal{A}, X^*)$, where $P' = P \circ Q|(L^1(\mathcal{A}, X^*))^{**} = L^1(\mathcal{A}, X^*)^{\perp\perp} \subset C([0, 1], X^*)^{***}$ and similarly $P'' : L^1(\mathcal{B}, X^*)^{**} \to L^1(\mathcal{B}, X^*)$ is defined by $P'' = E \circ P \circ Q|(L^1(\mathcal{B}, X^*))^{**}$. It is easy to see that $P' = P''$ on $L^1(\mathcal{B}, X^*)^{**}$.

Now let $f \in L^1(\mathcal{A}, X^*)$. We have $d(f, L^1(\mathcal{B}, X^*)^{\perp\perp}) = \|f - \Lambda\|$ for some $\Lambda \in L^1(\mathcal{B}, X^*)^{\perp\perp}$.

For any $g \in L^1(\mathcal{B}, X^*)$,

$$\|f - g\| \geq \|f - \Lambda\| \geq \left\|P'(f - \Lambda)\right\| = \left\|f - P''(\Lambda)\right\|.$$