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A topological approach to periodic oscillations related to the Liebau phenomenon

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ABSTRACT

We give some sufficient conditions for existence, non-existence and localization of positive solutions for a periodic boundary value problem related to the Liebau phenomenon. Our approach is of topological nature and relies on the Krasnosel'skiĭ–Guo theorem on cone expansion and compression. Our results improve and complement earlier ones in the literature.

1. Introduction

In the 1950s the physician G. Liebau developed some experiments dealing with a valveless pumping phenomenon arising in blood circulation and that has been known for a long time: roughly speaking, Liebau showed experimentally that a periodic compression made on an asymmetric part of a fluid-mechanical model could produce the circulation of the fluid without the necessity of a valve to ensure a preferential direction of the flow [1,12,13]. After his pioneering work this effect has been known as the Liebau phenomenon.

In [14] G. Propst, with the aim of contributing to the theoretical understanding of the Liebau phenomenon, presented some differential equations modeling a periodically forced flow through different pipe–tank configurations. He was able to prove the presence of pumping effects in some of them, but the apparently simplest model, the "one pipe–one tank" configuration, skipped his efforts due to a singularity in the corresponding differential equation model, namely

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$$\begin{cases} u''(t) + au'(t) = \dfrac{1}{u}\big(e(t) - b\big(u'(t)\big)^2\big) - c, & t \in [0,T], \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1}$$

being $a \geq 0$, $b > 1$, $c > 0$ and $e(t)$ continuous and T-periodic on $\mathbb{R}$.

The singular periodic problem (1) was studied in [4], where the authors gave general results for the existence and asymptotic stability of positive solutions by performing the change of variables $u = x^\mu$, where $\mu = \frac{1}{b+1}$, which transforms the singular problem (1) into the regular one

$$\begin{cases} x''(t) + ax'(t) = \dfrac{e(t)}{\mu}x^{1-2\mu}(t) - \dfrac{c}{\mu}x^{1-\mu}(t), & t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T). \end{cases} \tag{2}$$

Then the existence and stability of positive solutions for (2) were obtained by means of the lower and upper solution technique [5] and using tricks analogous to those used in [15].

In this paper we deal with the existence of positive solutions for the following generalization of the problem (2)

$$\begin{cases} x''(t) + ax'(t) = r(t)x^\alpha(t) - s(t)x^\beta(t), & t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{3}$$

where we assume

(H0) $a \geq 0$, $r, s \in \mathcal{C}[0,T]$, $0 < \alpha < \beta < 1$.

Note that, by defining $r(t) = \frac{e(t)}{\mu}$, $s(t) = \frac{c}{\mu}$, $\alpha = 1 - 2\mu$ and $\beta = 1 - \mu$, the problem (2) fits within (3).

Our approach is essentially of topological nature: in Section 2 we rewrite problem (3) as an equivalent fixed point problem suitable to be treated by means of the Krasnosel'skiĭ–Guo cone expansion/compression fixed point theorem. A careful analysis of the related Green's function, necessary in our approach, is postponed to a final Appendix A. In Section 3 we present our main results: existence, non-existence and localization criteria for positive solutions of the problem (3). Some corollaries with more ready-to-use results are also addressed. We point out that our results are valid not only for the more general problem (3), but also when applied to the singular model problem (1) we improve previous results of [4].

2. A fixed point formulation

First of all, by means of a shifting argument, we rewrite the problem (3) in the equivalent form

$$\begin{cases} x''(t) + ax'(t) + m^2x(t) = r(t)x^\alpha(t) - s(t)x^\beta(t) + m^2x(t) := f_m\big(t, x(t)\big), & t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases} \tag{4}$$

with $m \in \mathbb{R}$. A similar approach has been used, under a variety of boundary conditions in [7,9,16,17].

We say that problem (4) is non-resonant if zero is the unique solution of the homogeneous linear problem

$$\begin{cases} x''(t) + ax'(t) + m^2x(t) = 0, & t \in [0,T], \\ x(0) = x(T), \quad x'(0) = x'(T). \end{cases}$$

In this case the non-homogeneous linear problem