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Nonoscillation of all solutions of a higher order nonlinear delay dynamic equation on time scales

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A B S T R A C T

The authors give sufficient conditions for all solutions of a higher order nonlinear delay dynamic equation on time scales to be nonoscillatory. The results cover both the superlinear and sublinear cases and, in the differential equations case, resolve a question that has been open for 30 years. In order to prove their results, the authors first prove a new Bihari type inequality for functions on time scales.

1. Introduction

Beginning with the pioneering work of Sturm [25], who, many believed, proved his famous comparison theorem for difference equations before publishing it for differential equations (see Reid [18]), oscillation theory has attracted researchers for nearly two centuries. The 1955 publication of the seminal paper on second-order differential equations by Atkinson [2] initiated interest in the oscillatory behavior of solutions of nonlinear differential equations and for many years this has been one of the most active areas of mathematical research. Interest spread to higher order equations, equations with time delays, neutral equations, difference equations, and more recently to dynamic equations on time scales, impulsive equations, equations with fractional derivatives, stochastic equations, etc.

In the literal sense thousands of papers in the literature on such problems, finding sufficient conditions for all solutions of an equation to be oscillatory have been a major focus of study. Sufficient conditions

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for the existence of a nonoscillatory solution are more rare. In a number of papers in the 1970s, Kartsatos (see, for example, his very nice survey paper [15]) raised the question as to what types of forcing terms can maintain the oscillation of an already oscillatory unforced equation. This problem is occasionally revisited even today.

A much more difficult problem is to find sufficient conditions for all solutions of an equation to be nonoscillatory, and this is especially true for equations with delays; we will return to this point later. Of course, we are familiar with the condition of Hille [14] for the second order linear differential equation,

$$x'' + q(t)x = 0,$$

to be nonoscillatory, namely,

$$\int^{\infty} t\left|q(t)\right| dt < \infty.$$

Sufficient conditions for all solutions of nonlinear equations to be nonoscillatory are much more difficult to find because the techniques are often *ad-hoc* and do not extend well to more general equations or to equations of higher order. This is especially true when trying to find integral conditions as opposed to point-wise type conditions.

In this paper we consider the n-th order nonlinear delay dynamic equation

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta^{n-1}} + q(t)f\left(x\left(g(t)\right)\right) = r(t), \tag{1.1}$$

where $\mathbb{T}$ is a time scale with $\sup \mathbb{T} = \infty$, $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$, $a, q, r \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $g \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $f \in C(\mathbb{R},\mathbb{R})$, $a(t) > 0$, $g(t) \le t$, and $g(t) \to \infty$ as $t \to \infty$. Our goal is to provide sufficient conditions for all solutions of (1.1) to be nonoscillatory. Several flawed attempts have been made to obtain this kind of results for the case where Eq. (1.1) is a differential equation; for example, see [6–8, 20–24] and the discussion in [10]. As a consequence, correct results are only for solutions satisfying certain growth conditions and/or for the case where f is sublinear (see, for example, [11,12,22,24]). The superlinear case, $\gamma > 1$, was posed as an open problem in [10].

The approach we take here is to use the forcing term $r(t)$ to stop the oscillations. That is, the unforced equation may in fact have all or some of its solutions oscillatory, yet we will give conditions so that all solutions of the forced equation are nonoscillatory. Note the contrast to the results of Kartsatos mentioned above.

The usual notation and concepts from the time scale calculus as can be found in Bohner and Peterson [3,4] will be used throughout the paper without further mention.

2. Sufficient conditions for nonoscillation

Since we are working on a time scale, the notion of oscillation takes the form of what is known as a generalized zero of a function. We say that $x(t)$ has a *generalized zero* at a point T if $x(T)x(\sigma(T)) \le 0$, where $\sigma(t)$ is the usual forward jump operator. A function is said to be *oscillatory* if it has arbitrarily large generalized zeros and *nonoscillatory* otherwise.

We need to introduce the Taylor monomials $\{h_n(t,s)\}_{n=0}^{\infty}$ (see [3, §1.6]) that are defined recursively by

$$h_0(t,s) = 1 \quad \text{and} \quad h_{n+1}(t,s) = \int_s^t h_n(u,s)\Delta u, \quad n \ge 1. \tag{2.1}$$

If $n = 1$, then $h_1(t,s) = t - s$, but in general this expression does not hold for $n \ge 2$. However, if $\mathbb{T} = \mathbb{R}$, then $h_n(t,s) = (t-s)^n/n!$; if $\mathbb{T} = \mathbb{N}_0$, then $h_n(t,s) = (t-s)^{\underline{n}}/n!$, where $t^{\underline{n}} = t(t-1)\cdots(t-n+1)$ is the so-called