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# Chernoff's distribution and differential equations of parabolic and Airy type

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ABSTRACT

We give a direct derivation of the distribution of the maximum and the location of the maximum of one-sided and two-sided Brownian motion with a negative parabolic drift. The argument uses a relation between integrals of special functions, in particular involving integrals with respect to functions which can be called "incomplete Scorer functions". The relation is proved by showing that both integrals, as a function of two parameters, satisfy the same extended heat equation, and the maximum principle is used to show that these solutions must therefore have the stated relation. Once this relation is established, a direct derivation of the distribution of the maximum and location of the maximum of Brownian motion minus a parabola is possible, leading to a considerable shortening of the original proofs.



## 1. Introduction

Let $\{W(t) : t \in \mathbb{R}\}$ be a standard two-sided Brownian motion, originating from zero. The determination of the distribution of the (almost surely unique) location of the maximum of $\{W(t) - t^2 : t \in \mathbb{R}\}$ has a long history, which probably started with Chernoff's paper [1] in a study of the limit distribution of an estimator of the mode of a distribution. In the latter paper the density of the location of the maximum of $\{W(t) - t^2 : t \in \mathbb{R}\}$, which we will denote by

$$Z = \operatorname{argmax}_t \{W(t) - t^2,\ t \in \mathbb{R}\}, \tag{1.1}$$

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is characterized in the following way. Let $u(t,x)$ be the solution of the heat equation

$$\frac{\partial}{\partial t}u(t,x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x),$$

for $x \le t^2$, under the boundary conditions

$$u(t,x) \ge 0, \qquad u\bigl(t,t^2\bigr) \stackrel{\text{def}}{=} \lim_{x\uparrow t^2} u(t,x) = 1, \quad (t,x) \in \mathbb{R}^2, \qquad \lim_{x\downarrow -\infty} u(t,x) = 0, \quad t \in \mathbb{R}.$$

Furthermore, let the function $u_2$ be defined by

$$u_2(t) = \lim_{x\uparrow t^2} \frac{\partial}{\partial x}u(t,x).$$

Then the density of (1.1) is given by

$$f_Z(t) = \frac{1}{2}u_2(t)u_2(-t), \quad t \in \mathbb{R}. \tag{1.2}$$

The original attempts to compute the density $f_Z$ were based on numerically solving the heat equation above, but it soon became clear that this method did not produce a very accurate solution, mainly because of the rather awkward boundary conditions. However, around 1984 the connection with Airy functions was discovered and this connection was exploited to give analytic solutions in the papers [2,13,4], which were all written in 1984, although the last paper appeared much later.

There seems to be a recent revival of interest in this area of research, see, e.g., [9,5–7,10,8]. Also, the main theorem (Theorem 2.3 in [3]) uses Theorem 3.1 of [4] in an essential way. These recent papers (except [10]) rely a lot on the results in [2] and [4], but it seems fair to say that the derivation of these results in [2] and [4] is not a simple matter. The most natural approach still seems to use the Cameron–Martin–Girsanov formula for making the transition from Brownian motion with drift to Brownian motion without drift, and next to use the Feynman–Kac formula for determining the distribution of the Radon–Nikodym derivative of the Brownian motion with parabolic drift with respect to the Brownian motion without drift from the corresponding second order differential equation. This is the approach followed in [4]. However, the completion of these arguments used a lot of machinery which one would prefer to avoid. For this reason we give an alternative approach in the present paper.

The starting point of our approach is Theorem 2.1 in [4], which is given below for convenience. Theorem 2.1 in [4] in fact deals with the process $\{W(t) - ct^2 : t \in \mathbb{R}\}$ for an arbitrary positive constant $c > 0$, but since we can always deduce the results for general $c$ from the case $c = 1$, using Brownian scaling, see, e.g., [8], we take for convenience $c = 1$ in the theorem below. Another simplification is that we consider first hitting times of 0 for processes starting at $x < 0$ instead of first hitting times of $a$ of processes starting at $x < a$ for an arbitrary $a \in \mathbb{R}$, using space homogeneity. We made slight changes of notation, in particular the function $h_x$, $x > 0$, of [4] is again denoted by $h_x$, but now with a negative argument, so $h_x$ in our paper corresponds to $h_{-x}$ in [4].

**Theorem 1.1.** *(See Theorem 2.1 in [4].) Let, for $s \in \mathbb{R}$ and $x < 0$, $Q^{(s,x)}$ be the probability measure on the Borel $\sigma$-field of $C([s,\infty) : \mathbb{R})$, corresponding to the process $\{X(t) : t \ge s\}$, where $X(t) = W(t) - t^2$, starting at position $x$ at time $s$, and where $\{W(t) : t \ge s\}$ is Brownian motion, starting at $x + s^2$ at time $s$. Let the first passage time $\tau_0$ of the process $X$ be defined by*

$$\tau_0 = \inf\bigl\{t \ge s : X(t) = 0\bigr\},$$

*where, as usual, we define $\tau_0 = \infty$, if $\{t \ge s : X(t) = 0\} = \emptyset$. Then:*