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Extremal measures with prescribed moments

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A B S T R A C T

In the approximate integration some inequalities between the quadratures and the integrals approximated by them are called *extremalities*. On the other hand, the set of all quadratures is convex. We are trying to find possible connections between extremalities and extremal quadratures (in the sense of extreme points of a convex set). Of course, the quadratures are the integrals with respect to discrete measures and, moreover, a quadrature is extremal if and only if the associated measure is extremal. Hence the natural problem arises to give some description of extremal measures with prescribed moments in the general (not only discrete) case. In this paper we deal with symmetric measures with prescribed first four moments. The full description (with no symmetry assumptions, and/or not only four moments are prescribed and so on) is far to be done.

1. Introduction

The second-named author considered in [7] the so-called *extremalities* in the approximate integration. Let P_n be the n-th degree Legendre polynomial given by the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(x^2 - 1\right)^2.$$

Then P_n has n distinct roots $x_1, \ldots, x_n \in (-1, 1)$. The n-point Gauss–Legendre quadrature is the positive linear functional on $\mathbb{R}^{[-1,1]}$ given by

$$\mathcal{G}_n[f] = \sum_{i=1}^{n} w_i f(x_i)$$

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with the weights

$$w_i = \frac{2(1-x_i^2)}{(n+1)^2 P_{n+1}^2(x_i)}, \quad i = 1,\dots,n.$$

The $(n+1)$-point Lobatto quadrature is the functional

$$\mathcal{L}_{n+1}[f] = v_1 f(-1) + v_{n+1} f(1) + \sum_{i=2}^{n} v_i f(y_i),$$

where $y_2,\dots,y_n \in (-1,1)$ are (distinct) roots of P_n' and

$$v_1 = v_{n+1} = \frac{2}{n(n+1)}, \qquad v_i = \frac{2}{n(n+1)P_n^2(y_i)}, \quad i = 2,\dots,n.$$

For these forms of quadratures as well as for another quadratures appearing in this paper see for instance [2].

Recall that a continuous function $f : [-1,1] \to \mathbb{R}$ is *n-convex* ($n \in \mathbb{N}$), if and only if f is of the class $\mathcal{C}^{n-1}$ and the derivative $f^{(n-1)}$ is convex (cf. [4, Theorem 15.8.4]). For the needs of this paper it could be regarded as a definition of n-convexity.

Let $\mathcal{T}$ be a positive linear functional defined (at least) on a linear subspace of $\mathbb{R}^{[-1,1]}$ generated by the cone of $(2n-1)$-convex functions (i.e. $\mathcal{T}[f] \geqslant 0$ for $f \geqslant 0$). Assume that $\mathcal{T}$ is exact on polynomials of order $2n-1$, i.e. $\mathcal{T}[p] = \int_{-1}^{1} p(x)\mathrm{d}x$ for any polynomial p of order $2n-1$. It was proved in [7, Theorem 14] that the inequality

$$\mathcal{G}_n[f] \leqslant \mathcal{T}[f] \leqslant \mathcal{L}_{n+1}[f] \tag{1}$$

holds for any $(2n-1)$-convex function $f : [-1,1] \to \mathbb{R}$. Then the functionals $\mathcal{G}_n$ and $\mathcal{L}_{n+1}$ restricted to the cone of $(2n-1)$-convex functions are minimal and maximal, respectively, among all positive linear functionals defined (at least) on $(2n-1)$-convex functions, which are exact on polynomials of order $2n-1$. In [7, Theorem 15] there is a counterpart of the above result for $2n$-convex functions with Radau quadratures in the role of the minimal and maximal operators.

Studying the results of this kind the following problem seems to be natural. Some quadrature operators are extremal in the sense of inequalities like (1). On the other hand, the set of all quadratures which are exact on polynomials of some given order is convex. Then it could be interesting to find its extreme points looking for the possible connections between extremalities in the approximate integration and the extreme points of convex sets. In particular, are $\mathcal{G}_n$ and $\mathcal{L}_{n+1}$ extreme points of the above mentioned set? If the answer is positive, are they the only extreme points, or do there exist another ones?

This is the starting point for our considerations. We will observe that the extreme points in the set of all quadratures exact on polynomials of prescribed order could be determined with the aid of [3, Theorem 6.1, p. 101]. Next we shall investigate the extreme points of the set of all positive linear operators defined on $\mathcal{C}[-1,1]$ with prescribed moments. Our research is far from being complete. Actually we are able to give a full description of the extreme points of the set of symmetric operators with four prescribed moments, i.e. $(m_0, m_1, m_2, m_3) = (1, 0, b^2, 0)$.

2. Extremal quadratures

Let D be a convex subset of a linear space. Recall that $x \in D$ is the extreme point of D, if x is not the "interior" point of any segment with endpoints in D, i.e. $x = tu + (1-t)v$ for some $u, v \in D$ and $t \in [0,1]$ implies that $x = u = v$. The set of all extreme points of a set D will be denoted by $\operatorname{ext} D$.