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Weighted eigenvalue problems for quasilinear elliptic operators with mixed Robin–Dirichlet boundary conditions

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ABSTRACT

We investigate the existence of principal eigenvalues type problems with weights for the quasilinear operator $-\Delta_p + V\psi_p$ with mixed weighted Robin–Dirichlet boundary conditions in a bounded regular domain. We also give some results on the existence of nonprincipal eigenvalues.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and let ν be its outer normal defined everywhere. Let V be a bounded function defined in Ω and σ a smooth function defined on $\partial\Omega$. It was pointed out in [9,8] that the Fourier analysis for parabolic problems with dynamic boundary conditions leads, through a separation of variables, to the following eigenvalue problem with Robin type boundary conditions

$$\begin{cases} -\Delta u + V(x)u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda \sigma(x) u & \text{on } \partial\Omega. \end{cases} \tag{L}$$

The complete analysis of this eigenvalue problem when $V \geq 0$ has been done in [9] in the case $\sigma = cst$, and in [8] in the case $\sigma \neq cst$ (for a slighter general operator than the laplacian). As a common feature, it appears that problem (L) possesses an infinite sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ if σ^+ (the positive part of σ) is $\not\equiv 0$, and an infinite sequence of negative eigenvalues $\{\lambda_{-n}\}_{n\in\mathbb{N}}$ if $\sigma^- \not\equiv 0$ and $N \geq 2$. Moreover,

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$\lambda_{\pm 1}$ are both principal eigenvalue (i.e., an eigenvalue whose eigenfunctions are sign-constant) and simple (i.e. the associated eigenfunctions are each a constant multiple of one another).

It is also well known that the spectra of the $-\Delta + V$, in the case $V^- \not\equiv 0$, could present features different from those of the spectra in the case $V \geq 0$. Problem (L) with V indefinite and Dirichlet boundary conditions has been extensively studied for instance by Allegretto–Mingarelli [4], Fleckinger–Hernandez–de Thelin [15] and Lopez-Gomez [20] among others.

Our intention in this paper is to initiate the *study of the spectrum* of the more general problem

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda \sigma |u|^{p-2}u & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2, \end{cases} \tag{P}$$

with V, m and σ indefinite. Here $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, denotes the p-Laplacian operator for $p > 1$. We will assume that Ω is a bounded smooth domain and that $\partial\Omega$ splits up in two sets Γ_1 and Γ_2 which are connected and closed $(n-1)$-manifolds.

The existence of principal eigenvalues for the quasilinear equation in problem (P) with Dirichlet boundary condition and V, m indefinite has been treated by Binding–Huang [10,11] and [13]. It appears that sometimes there are not principal eigenvalues, a phenomenon that depends, loosely speaking, on how big the negative part of V with respect to the negative part of m is.

Problem (P) for $p = 2$ with $V \equiv 0$, $\Gamma_2 = \emptyset$ and m indefinite has already been considered by Afrouzi–Brown [2] when $\sigma = cst$, and later by K. Umezu [23] for indefinite σ. This last author proved that, besides the trivial eigenvalue $\lambda = 0$, problem (P) possesses a *unique* positive principal eigenvalue if and only if

$$\oint_{\partial\Omega} \sigma \, d\rho + \int_{\Omega} m \, dx < 0.$$

Here $d\rho$ stands for the surface element of $\partial\Omega$.

In the case $V \not\equiv 0$ and possibly indefinite, the situation is much different since the energy functional $E_V(u) \stackrel{\text{def}}{=} \int_\Omega (|\nabla u|^p + V|u|^p)dx$ is indefinite. One approach to find principal eigenvalues that has been used by many authors is to define a new eigenvalue problem for each fixed λ and to construct "an eigenvalue curve" as λ varies. We apply this approach in Section 3 and we give in Section 4 a necessary and sufficient condition for the existence of eigenvalues in terms of the infimum of E_V over the set of functions $\mathcal{G}$ satisfying $\int_\Omega |u|^p \, dx = 1$ and

$$\int_{\Omega} m|u|^p \, dx + \oint_{\Gamma_1} |u|^p \sigma \, d\rho = 0.$$

Indeed, this set $\mathcal{G}$ was already considered in [10,11] and [13] for the quasilinear equation of problem (P) with Dirichlet, Newmann or mixed boundary conditions. In fact, one cannot exclude that some eigenfunctions belong to $\mathcal{G}$ and, in that case, the corresponding energy levels are called in [11] "ghost states". Ghost states are interesting because they have the property of *losing of compactness*, see the discussion in Section 8 and Remark 8.6.

This paper is organized as follows. We construct the eigencurve associated to problem (P) in Section 3. The existence of principal eigenvalues is studied in Section 4. A sufficient condition for the existence of principal eigenvalues is presented in Section 5, where we also discuss the necessity of such condition for the coerciveness of the related functional. In Section 6 we prove isolation and simplicity of principal eigenvalues of problem (P). In Section 7 we investigate the coerciveness of the restricted functional and in Section 8 we prove the existence of two unbounded sequences of eigenvalues in the case where either $m^\pm \not\equiv 0$ or $\sigma^\pm \not\equiv 0$.