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On an identity for zeros of Bessel functions

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ABSTRACT

In this paper our aim is to present an elementary proof of an identity of Calogero concerning the zeros of Bessel functions of the first kind. Moreover, by using our elementary approach we present a new identity for the zeros of Bessel functions of the first kind, which in particular reduces to some other new identities. We also show that our method can be applied for the zeros of other special functions, like Struve functions of the first kind, and modified Bessel functions of the second kind.

1. Introduction and main results

In 1977 F. Calogero [3] deduced the following identity

$$\sum_{n\geq 1,\, n\neq k} \frac{1}{j_{\nu,n}^2 - j_{\nu,k}^2} = \frac{\nu+1}{2j_{\nu,k}^2}, \tag{1.1}$$

where $\nu > -1$, $k \in \{1, 2, \ldots\}$ and $j_{\nu,n}$ stands for the nth positive zero of the Bessel function of the first kind J_ν. Calogero's proof [3] of (1.1) is based on the infinite product representation of the Bessel functions of the first kind and on the clever use of an equivalent form of the Mittag-Leffler expansion

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$$\frac{J_{\nu+1}(x)}{J_\nu(x)} = \sum_{n\geq 1} \frac{2x}{j_{\nu,n}^2 - x^2}, \tag{1.2}$$

where $\nu > -1$. Note that in [3] it was pointed out that results like (1.1) are related to the connection between the motion of poles and zeros of special solutions of partial differential equations and many-body problems. In 1986 Ismail and Muldoon [8] mentioned that (1.1) can be obtained also by evaluating the residues of the functions in (1.1) at their poles. For other results on series of zeros of Bessel functions of the first kind and other special functions we suggest to the reader to refer [1,10]. In this paper our aim is to present an alternative proof of (1.1) by using only elementary analysis. Our proof is based on the Mittag-Leffler expansion (1.2), recurrence relations, the Bessel differential equation and on the Bernoulli–L'Hospital rule for the limit of quotients. Moreover, by using our idea we are able to prove the following new results.

Theorem 1. *If $\nu > -1$ and $k \in \{1, 2, \ldots\}$, then we have*

$$\sum_{n\geq 1,\, n\neq k} \frac{1}{j_{\nu,n}^4 - j_{\nu,k}^4} = -\frac{1}{2j_{\nu,k}^2} \sum_{n\geq 1} \frac{1}{j_{\nu,n}^2 + j_{\nu,k}^2} + \frac{\nu+2}{4j_{\nu,k}^4}. \tag{1.3}$$

In particular, for all $k \in \{1, 2, \ldots\}$ we have

$$\sum_{n\geq 1,\, n\neq k} \frac{1}{n^4 - k^4} = -\frac{\pi}{4k^3}\coth(k\pi) + \frac{7}{8k^4}. \tag{1.4}$$

Moreover, for all $k \in \{1, 3, \ldots\}$ we have

$$\sum_{\substack{n\geq 1,\, n\neq k \\ n \text{ is odd}}} \frac{1}{n^4 - k^4} = -\frac{\pi}{8k^3}\tanh\left(\frac{k\pi}{2}\right) + \frac{3}{8k^4}. \tag{1.5}$$

As far as we know the above results are new and as we can see below our method can be applied for the zeros of other special functions, like Struve functions and modified Bessel functions of the second kind. Our proof in this case is based on the corresponding Mittag-Leffler expansion for Struve functions, recurrence relations, the Struve differential equation and on the Bernoulli–L'Hospital rule. During the process of writing this paper Michael Milgram informed us that the relation (1.4) can be found in Hansen's book [7, p. 108]. Moreover, an alternative proof of (1.3) was proposed by Christophe Vignat. His proof was based on the formula

$$\sum_{n\geq 1,\, n\neq k} \frac{1}{j_{\nu,n}^4 - j_{\nu,k}^4} = \frac{1}{2j_{\nu,k}^2} \sum_{n\geq 1,\, n\neq k} \frac{1}{j_{\nu,n}^2 - j_{\nu,k}^2} - \frac{1}{2j_{\nu,k}^2} \sum_{n\geq 1,\, n\neq k} \frac{1}{j_{\nu,n}^2 + j_{\nu,k}^2}, \tag{1.6}$$

which by means of (1.1) evidently implies (1.3). It is worth to mention that (1.6) holds true in fact for any set of numbers $\{j_{\nu,n}\}_{n\geq 1}$, and by using this idea we can get (1.8) for the zeros of Struve functions.

Theorem 2. *Let $h_{\nu,n}$ be the nth positive zero of the Struve function of the first kind $\mathbf{H}_\nu$. If $|\nu| < \frac{1}{2}$ and $k \in \{1, 2, \ldots\}$, then the following identities are valid*

$$\sum_{n\geq 1,\, n\neq k} \frac{1}{h_{\nu,n}^2 - h_{\nu,k}^2} = \frac{\nu+2}{2h_{\nu,k}^2} - \frac{h_{\nu,k}^{\nu-2}}{\sqrt{\pi}2^{\nu-1}\Gamma(\nu+\frac{1}{2})\mathbf{H}_\nu'(h_{\nu,k})}, \tag{1.7}$$

$$\sum_{n\geq 1,\, n\neq k} \frac{1}{h_{\nu,n}^4 - h_{\nu,k}^4} = -\frac{1}{2h_{\nu,k}^2} \sum_{n\geq 1} \frac{1}{h_{\nu,n}^2 + h_{\nu,k}^2} + \frac{\nu+3}{4h_{\nu,k}^4} - \frac{h_{\nu,k}^{\nu-4}}{\sqrt{\pi}2^{\nu}\Gamma(\nu+\frac{1}{2})\mathbf{H}_\nu'(h_{\nu,k})}. \tag{1.8}$$