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Real analytic families of harmonic functions in a planar domain with a small hole

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ABSTRACT

We consider a Dirichlet problem in a planar domain with a hole of diameter proportional to a real parameter ϵ and we denote by u_ϵ the corresponding solution. The behavior of u_ϵ for ϵ small and positive can be described in terms of real analytic functions of two variables evaluated at $(\epsilon, 1/\log\epsilon)$. We show that under suitable assumptions on the geometry and on the boundary data one can get rid of the logarithmic behavior displayed by u_ϵ for ϵ small and describe u_ϵ by real analytic functions of ϵ. Then it is natural to ask what happens when ϵ is negative. The case of boundary data depending on ϵ is also considered. The aim is to study real analytic families of harmonic functions which are not necessarily solutions of a particular boundary value problem.

1. Introduction

This paper continues the work begun by the authors in [1]. Indeed, in [1], the case of harmonic function in a perforated domain of $\mathbb{R}^n$, with $n \geq 3$, has been investigated. Here instead we focus on the two-dimensional case. We begin by introducing some notation. We fix once for all

$$\alpha \in]0,1[.$$

Then we fix two sets Ω^o and Ω^i in the two-dimensional Euclidean space $\mathbb{R}^2$. The letter 'o' stands for 'outer domain' and the letter 'i' stands for 'inner domain'. We assume that Ω^o and Ω^i satisfy the following condition.

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$$\begin{aligned}&\Omega^o \text{ and } \Omega^i \text{ are open bounded connected subsets of } \mathbb{R}^2 \text{ of}\\&\text{class } C^{1,\alpha} \text{ such that } \mathbb{R}^2 \setminus \mathrm{cl}\Omega^o \text{ and } \mathbb{R}^2 \setminus \mathrm{cl}\Omega^i \text{ are connected}\\&\text{and such that the origin } 0 \text{ of } \mathbb{R}^2 \text{ belongs both to } \Omega^o \text{ and } \Omega^i.\end{aligned} \tag{1}$$

Here and in the sequel cl denotes the closure. For the definition of functions and sets of the usual Schauder classes $C^{0,\alpha}$ and $C^{1,\alpha}$, we refer for example to Gilbarg and Trudinger [5, §6.2]. We note that condition (1) implies that Ω^o and Ω^i have no holes and that there exists a real number ϵ_0 such that

$$\epsilon_0 > 0 \quad \text{and} \quad \epsilon\,\mathrm{cl}\,\Omega^i \subseteq \Omega^o \quad \text{for all } \epsilon \in \,]-\epsilon_0, \epsilon_0[.$$

Then we denote by $\Omega(\epsilon)$ the perforated domain defined by

$$\Omega(\epsilon) \equiv \Omega^o \setminus \left(\epsilon\,\mathrm{cl}\,\Omega^i\right) \quad \forall \epsilon \in \,]-\epsilon_0, \epsilon_0[.$$

A simple topological argument shows that $\Omega(\epsilon)$ is an open bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ for all $\epsilon \in \,]-\epsilon_0, \epsilon_0[\, \setminus \{0\}$. Moreover, the boundary $\partial\Omega(\epsilon)$ of $\Omega(\epsilon)$ has exactly the two connected components $\partial\Omega^o$ and $\epsilon\partial\Omega^i$, for all $\epsilon \in \,]-\epsilon_0, \epsilon_0[$. We also note that $\Omega(0) = \Omega^o \setminus \{0\}$.

Now let $g^o \in C^{1,\alpha}(\partial\Omega^o)$ and $g^i \in C^{1,\alpha}(\partial\Omega^i)$. For all $\epsilon \in \,]-\epsilon_0, \epsilon_0[\, \setminus \{0\}$, let u_ϵ be the unique function of $C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon))$ such that

$$\begin{cases} \Delta u_\epsilon = 0 & \text{in } \Omega(\epsilon), \\ u_\epsilon(x) = g^o(x) & \text{for } x \in \partial\Omega^o, \\ u_\epsilon(x) = g^i(x/\epsilon) & \text{for } x \in \epsilon\partial\Omega^i. \end{cases} \tag{2}$$

Let u_0 be the unique function of $C^{1,\alpha}(\mathrm{cl}\,\Omega^o)$ such that

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega^o, \\ u_0(x) = g^o(x) & \text{for } x \in \partial\Omega^o. \end{cases} \tag{3}$$

We fix a point p in $\Omega^o \setminus \{0\}$ and take $\epsilon_p \in \,]0, \epsilon_0[$ such that $p \in \Omega(\epsilon)$ for all $\epsilon \in \,]-\epsilon_p, \epsilon_p[$. Then $u_\epsilon(p)$ is defined for all $\epsilon \in \,]-\epsilon_p, \epsilon_p[$ and we can ask, for example, the following question.

What can be said of the function from $]0, \epsilon_p[$ to $\mathbb{R}$ which takes ϵ to $u_\epsilon(p)$?

Questions of this type are typical in the frame of asymptotic analysis and are usually investigated by means of asymptotic expansion methods (see for example Maz'ya, Nazarov, and Plamenevskij [11, §2.4.1]). The techniques of asymptotic analysis usually aim at representing the behavior of $u_\epsilon(p)$ as $\epsilon \to 0^+$ in terms of regular functions of ϵ plus a remainder which is smaller than a known infinitesimal function of ϵ. In this paper, instead, we adopt the functional analytic approach proposed by Lanza de Cristoforis. By such an approach, one can prove that there exist $\epsilon_p \in \,]0, \epsilon_0]$, $\epsilon_p < 1$, and a real analytic function U_p from $]-\epsilon_p, \epsilon_p[\, \times \,]1/\log\epsilon_p, -1/\log\epsilon_p[$ to $\mathbb{R}$ such that

$$u_\epsilon(p) = U_p[\epsilon, 1/\log\epsilon] \quad \forall \epsilon \in \,]0, \epsilon_p[\tag{4}$$

and that $u_0(p) = U_p[0, 0]$ (cf., *e.g.*, Lanza de Cristoforis [9]). We observe that the logarithmic behavior displayed by u_ϵ for ϵ small only arises in dimension two and does not appear in higher dimensions (cf., *e.g.*, Lanza de Cristoforis [9]). Also, if instead of considering a Dirichlet boundary value problem we considered a mixed boundary value problem with a Dirichlet condition in the inner component of the boundary and a Neumann condition in the outer component, then one can prove that the logarithmic behavior appears