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Attractors of generalized IFSs that are not attractors of IFSs

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ABSTRACT

Mihail and Miculescu introduced the notion of a generalized iterated function system (GIFS in short), and proved that every GIFS generates an attractor. (In our previous paper we gave this notion a more general setting.) In this paper we show that for any $m \geq 2$, there exists a Cantor subset of the plane which is an attractor of some GIFS of order m, but is not an attractor of a GIFS of order $m-1$. In particular, this result shows that there is a subset of the plane which is an attractor of some GIFS, but is not an attractor of an IFS. We also give an example of a Cantor set which is not an attractor of a GIFS.

1. Introduction

Mihail and Miculescu in [12,14,15] introduced the notion of a generalized iterated function system (GIFS in short), which is a natural generalization of a classical IFS [8,1] (instead of selfmappings of a metric space X, GIFS consists of mappings $f : X^m \to X$, where X^m is the Cartesian product of m copies of X), and proved that every GIFS generates an attractor. In [19] we gave this notion a more general setting and strengthened (and gathered) most results of mentioned papers. In papers [11,13,16,18,20] the reader can find further studies on this notion.

A natural question arises, whether the notion of GIFS gives us something really new, i.e., whether there is a set which is an attractor of some GIFS, but is not an attractor of an IFS. Partial answers were given in [15, Example 4.3] and [19, Theorem 4.3]. However, sets constructed there did not solve this problem completely and, moreover, they are subsets of infinite dimensional space.

In this paper we solve this problem completely by constructing appropriate Cantor subsets of the plane.

We also give an example of a Cantor set on the plane which is not an attractor of a GIFS.

The construction and some ideas base on the paper of Crovisier and Rams [5].

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2. Definitions and basic notation

Let (X,d) and (Y,ρ) be metric spaces. If $f : X \to Y$, then by $Lip(f)$ we denote the Lipschitz constant of f (in the following, $\inf\emptyset = \infty$):

$$Lip(f) := \inf\bigl\{c > 0 : \forall_{x,y\in X}\ \rho\bigl(f(x), f(y)\bigr) \le cd(x,y)\bigr\}.$$

If $Lip(f) \le 1$, then we say that f is *nonexpansive*.

If for some function $\varphi : [0,\infty) \to [0,\infty)$, we have

$$\forall_{x,y\in X}\quad \rho\bigl(f(x), f(y)\bigr) \le \varphi\bigl(d(x,y)\bigr), \tag{1}$$

then we say that f is a *generalized φ-contraction*.

If $f : X \to Y$ is a generalized φ-contraction for a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ such that for every $t > 0$, $\varphi^k(t) \to 0$ (φ^k is the kth iteration of φ), then we say that f is *a generalized Matkowski contraction*.

Clearly, if $Lip(f) < 1$, then f is a generalized Matkowski contraction (simply set $\varphi(t) := tLip(f)$).

If f is a generalized Matkowski contraction, then for each $x, y \in X$ with $x \ne y$, we have

$$\rho\bigl(f(x), f(y)\bigr) < d(x,y). \tag{2}$$

Indeed, this follows from the fact that if φ is nondecreasing and $\varphi^k(t) \to 0$, then $\varphi(t) < t$.

Hence if X is compact, then

$$\forall_{t>0}\ \exists_{\alpha<1}\ \forall_{x,y\in X}\quad \bigl(d(x,y) \ge t \Rightarrow \rho\bigl(f(x), f(y)\bigr) \le \alpha d(x,y)\bigr). \tag{3}$$

It is easy to see that a function $f : X \to Y$ which satisfies (3) is a generalized Matkowski contraction. It follows that if X is compact and $f : X \to Y$, then f is a generalized Matkowski contraction iff f satisfies (2) iff f satisfies (3).

If $Y = X$, then a generalized Matkowski contraction is called a *Matkowski contraction* [10,9], and it is known that each Matkowski contraction on a complete metric space satisfies the thesis of the Banach fixed point theorem.

Remark 1. In the literature, many types of φ-contractions are considered. For example, it can be easily shown that a function $f : X \to X$ satisfies (3) iff f is a φ-contraction for some continuous function $\varphi : [0,\infty) \to [0,\infty)$ such that for any $t > 0$, $\frac{\varphi(t)}{t} < 1$ and the function $(0,\infty) \ni t \to \frac{\varphi(t)}{t}$ is nonincreasing. In this case we call such a function *a Rakotch contraction* [17]. Also, if $f : X \to X$ is a φ-contraction for a nondecreasing, upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) < t$ for $t > 0$, then f is called *a Browder contraction* [4].

The relations between these notions are as follows: Rakotch contractions are Browder contractions, and Browder contractions are Matkowski contractions (because if the function φ witnesses to the fact that f is a Browder contraction, then we must have $\varphi^k(t) \to 0$ for $t > 0$).

Finally, it is worth to mention that functions $f : X \to X$ which satisfy (2) are called *Edelstein contractions*, and the earlier observation shows that among compact spaces, the notions of Edelstein, Matkowski, Browder and Rakotch contractions coincide. For a deep discussion on various types of contractive conditions we refer the reader to [9].

By $\mathbf{K}(X)$ we denote the family of nonempty and compact subsets of X, considered as a metric space with the Hausdorff metric.