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Blow-up of smooth solutions to the compressible magnetohydrodynamic flows

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ABSTRACT

In this paper, we prove the blow-up phenomena of smooth solutions to the Cauchy problem for the full compressible magnetohydrodynamic equations and isentropic compressible magnetohydrodynamic equations with constant and degenerate viscosities under some restrictions on the initial data. In particular, our results do not require that the initial data have compact support or contain vacuum in any finite region.

1. Introduction

For its importance and application in industry and scientific research, the viscous compressible magnetohydrodynamic model has been studied both from a theoretical and numerical perspective. The governing equations for compressible magnetohydrodynamic flows in the Eulerian coordinates have the following form:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u + \operatorname{curl} B \times B, \\ \partial_t(\rho e) + \operatorname{div}(\rho e u) + p \operatorname{div} u = 2\mu\left|\mathfrak{D}(u)\right|^2 + \lambda(\operatorname{div} u)^2 + \kappa\Delta\theta + \nu|\operatorname{curl} B|^2, \\ \partial_t B - \operatorname{curl}(u\times B) = -\operatorname{curl}(\nu \operatorname{curl} B), \quad \operatorname{div} B = 0. \end{cases} \tag{1.1}$$

Here $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$ and $\rho = \rho(x,t)$, $u = (u_1, u_2, \cdots, u_n)$, θ, p, e and B denote the density, velocity, absolute temperature, pressure, internal energy and magnetic field, respectively. $\mathfrak{D}$ is given by

$$\mathfrak{D}(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^t\right),$$

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where I is the identity matrix, μ and λ are the coefficients of viscosity and the second coefficient of viscosity, respectively, which satisfy

$$\mu \geq 0, \qquad 2\mu + n\lambda \geq 0.$$

The coefficients ν and $\mathcal{K} \geq 0$ denote the magnetic diffusivity coefficient and the coefficient of heat conduction, respectively. According to thermodynamic law, the state equations satisfy the following form

$$p = R\rho\theta, \qquad e = c_\nu\theta, \qquad p = A\exp\left(\frac{s}{c_\nu}\right)\rho^\gamma, \tag{1.2}$$

where $R > 0$ is the gas constant, $A > 0$ is an absolute constant, $\gamma > 1$ is the specific heat ratio, $c_\nu = \frac{R}{\gamma-1}$ and s is the entropy. The pressure can be expressed as

$$p = (\gamma - 1)\rho e. \tag{1.3}$$

We supplement the system (1.1) with the initial data

$$(\rho, u, s, B)(x,t)\big|_{t=0} = \big(\rho_0(x), u_0(x), s_0(x), B_0(x)\big). \tag{1.4}$$

If the entropy is a constant, the full magnetohydrodynamic equations reduce to isentropic case

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u + \mathrm{curl}B \times B, \\ \partial_t B - \mathrm{curl}(u \times B) = -\mathrm{curl}(\nu\mathrm{curl}B), \quad \mathrm{div}B = 0. \end{cases} \tag{1.5}$$

The state equation of the isentropic process becomes

$$p = \rho^\gamma, \quad \gamma > 1. \tag{1.6}$$

The initial data to the equations are imposed as

$$(\rho, u, B)(x,t)\big|_{t=0} = \big(\rho_0(x), u_0(x), B_0(x)\big). \tag{1.7}$$

The compressible magnetohydrodynamic equations with coefficients depending on density is another kind of important model.

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla\rho^\gamma = \mathrm{div}\left[h(\rho)\left(\dfrac{\nabla u + \nabla^t u}{2}\right)\right] + \nabla\big(g(\rho)\mathrm{div}u\big) + \mathrm{curl}B \times B, \\ \partial_t B - \mathrm{curl}(u \times B) = -\mathrm{curl}\big(\nu(\rho)\mathrm{curl}B\big), \quad \mathrm{div}B = 0. \end{cases} \tag{1.8}$$

In particular, we consider the case $h(\rho) = \rho^\alpha$, $g(\rho) = (\alpha-1)\rho^\alpha$ and $\nu(\rho) = \rho^\beta$, where $\alpha > 1 - \frac{1}{n}$ and $\beta \geq 0$.

In fluid mechanics, the existence, uniqueness and blow-up have been the subject of many theoretical studies in recent years. Especially, there are extensive literatures on the blow-up phenomena. More precisely, Sideris [17] showed the life span of C^1 solution to the compressible Euler equations was finite when the initial data are constant outside a bounded set and the initial flow velocity has compact supports. In 1998, Xin [20] used a different method to prove the blow-up result for the compressible Navier–Stokes equations, under two basic hypotheses: the support of the density grows sublinearly in time and the entropy is bounded