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Uniform attractors for three-dimensional Navier–Stokes equations with nonlinear damping ☆

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ABSTRACT

This paper is concerned with the three-dimensional non-autonomous Navier–Stokes equation with nonlinear damping in 3D bounded domains. When the external force $f_0(x,t)$ is translation compact in $L^2_{\mathrm{loc}}(\mathbb{R};H)$, $\alpha > 0$, $\frac{7}{2} \le \beta \le 5$ and initial data $u_\tau \in V$, we give a series of uniform estimates on the solutions. Based on these estimates, we prove the family of processes $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$, is $(V \times \mathcal{H}(f_0), V)$-continuous. At the same time, by making use of Ascoli–Arzela theorem, we find $\{U_f(t,\tau)\}$, $f \in \mathcal{H}(f_0)$, is $(V, \mathbf{H}^2(\Omega))$-uniformly compact. So, using semiprocess theory, we obtain the existence of (V,V)-uniform attractor and $(V, \mathbf{H}^2(\Omega))$-uniform attractor. And we prove the (V,V)-uniform attractor is actually the $(V, \mathbf{H}^2(\Omega))$-uniform attractor.

1. Introduction

In the last years, also in the present, there has been a great interest in studying the asymptotic behavior of the weak solutions of the three-dimensional Navier–Stokes equations [7,4,11,2,3,10,9]. However, the uniqueness of weak solutions and the global existence of strong solutions for three-dimensional Navier–Stokes equation remain completely open. In [5], Xiaojing Cai and Quansen Jiu have investigated the existence and regularity of solutions for three-dimensional Navier–Stokes equation with nonlinear damping. And in [13], we have discussed the existence of global attractors for the strong solutions of three-dimensional Navier–Stokes equation with nonlinear damping. So far, there are no other results on the asymptotic behavior of solutions for this equation.

In this paper, we consider the following non-autonomous three-dimensional Navier–Stokes equation with nonlinear damping:

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$$\begin{cases} u_t - \mu\Delta u + (u\cdot\nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f(x,t), & x\in\Omega, t>\tau,\\ \operatorname{div} u = 0, & x\in\Omega, t>\tau,\\ u|_{t=\tau} = u_\tau, & x\in\Omega,\\ u|_{\partial\Omega} = 0, & t>\tau, \end{cases} \tag{1}$$

where $\mu > 0$ is the kinematic viscosity of the fluid and $f(x,t)$ is the external body force. $\Omega \subset \mathbb{R}^3$ is an open bounded set with the boundary $\partial\Omega$ smooth enough. The unknown functions here are $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$, which stand for the velocity field and the pressure of the flow, respectively. In dampness term, $\beta \geq 1$ and $\alpha > 0$ are two constants. The given function $u_\tau = u_\tau(x)$ is the initial velocity.

The mathematical framework of (1) is classical. We define the usual function spaces

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}, \qquad H = cl_{(L^2(\Omega))^3}\mathcal{V}, \qquad V = cl_{(H_0^1(\Omega))^3}\mathcal{V},$$

where cl_X denotes the closure in the space X. It is well known that H, V are separable Hilbert spaces and identifying H and its dual H', we have $V \hookrightarrow H \hookrightarrow V'$ with dense and continuous injections, and $V \hookrightarrow H$ is compact. H and V endowed, respectively, with the inner products

$$(u,v) = \int_\Omega u\cdot v \mathrm{d}x, \quad \forall u,v\in H,$$

$$((u,v)) = \sum_{i=1}^3 \int_\Omega \nabla u_i \cdot \nabla v_i \mathrm{d}x, \quad \forall u,v \in V,$$

and norms $|\cdot|_2 = (\cdot,\cdot)^{\frac{1}{2}}$, $\|\cdot\| = ((\cdot,\cdot))^{\frac{1}{2}}$. In this paper, $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, and we use $|\cdot|_p$ to denote the norm in $\mathbf{L}^p(\Omega)$.

If $u \in L^\infty(\tau,T;H) \cap L^2(\tau,T;V) \cap L^{\beta+1}(\tau,T;\mathbf{L}^{\beta+1}(\Omega))$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(u,v) + \mu((u,v)) + b(u,u,v) + (\alpha|u|^{\beta-1}u, v) = (f,v), & \forall v\in V, \forall t>\tau,\\ u(\tau) = u_\tau, \end{cases} \tag{2}$$

then we say that u is a weak solution of (1) on $[\tau, T]$.

The weak formulation (2) is equivalent to the function equation

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + \mu Au + B(u) + G(u) = f(x,t), & \text{for } t>\tau,\\ u(\tau) = u_\tau, \end{cases} \tag{3}$$

where $Au = -\tilde{P}\Delta u$ is the Stokes operator defined by $\langle Au, v\rangle = ((u,v))$, and $\tilde{P}$ is the orthogonal projection of $(L^2(\Omega))^3$ onto H. $G(u) = \tilde{P}F(u)$, and $F(u) = \alpha|u|^{\beta-1}u$. $B: V\times V \to V'$ is a bilinear operator defined by $\langle B(u,v), w\rangle = b(u,v,w)$, $B(u) = B(u,u)$, where

$$b(u,v,w) = \sum_{i=1}^3 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x,$$

and $\langle\cdot,\cdot\rangle$ is the duality product between V and V'.

In this paper, $\hookrightarrow$ denotes embedding between spaces, $\operatorname{dist}_E(X,Y) = \sup_{x\in X}\inf_{y\in Y} \operatorname{dist}(x,y)$ denotes the Hausdorff semi-distance between $X\subset E$ and $Y \subset E$ in the metric space E. C and $C(\cdot)$ denote the generic constant that may take different values in different places.