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Infinite time gradient blow-up for parabolic prescribed mean curvature equations

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A B S T R A C T

We prove some results on the existence of infinite time gradient blow-up phenomena for parabolic prescribed mean curvature equations over bounded, mean-convex domains in R^n.

1. Introduction

Consider the following quasilinear parabolic problem

$$\begin{cases} u_t - \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, mean-convex domain of class $C^{2+\alpha}$. Assume that $\varphi \in C^{2+\alpha}(\bar{\Omega})$ is compatible with the boundary condition $u = 0$. Here, $\alpha \in (0,1)$, $C^{k+\alpha}(\bar{\Omega})$ and $C^{k+\alpha,\frac{k+\alpha}{2}}(\bar{\Omega}\times[0,T])$ denote the standard Hölder spaces on $\bar{\Omega}$ and $\bar{\Omega}\times[0,T]$, "mean-convex" means $H' \geqslant 0$, where H' denotes the mean curvature of $\partial\Omega$. Notice that mean-convexity is weaker than convexity when $n \geqslant 3$.

We are interested in (1.1) with the following three types of f:

(A) $f(u) = \lambda u$,
(B) $f(u) = \lambda u - \sigma u^p$, $\sigma > 0$, $p > 1$,
(C) $f(u) = \lambda(u - u^p)$, $p > 1$,

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where λ is a positive parameter. These problems are of both mathematical and physical interest. Case (A) arises in the model of pendent liquid drops in capillary surfaces (see [5,11]). The solutions in cases (A) and (B) are also viewed as heat flows with the unit heat capacity, the conductivity $(1+|\nabla u|^2)^{-\frac{1}{2}}$, and the heat source f (see [4]). Case (C) is related to a phase transition model with large spatial gradients (see [1,2] for $p=3$).

In the present paper, we are concerned with the so-called infinite time gradient blow-up phenomenon for (1.1). Let $u(x,t)$ be a global solution of (1.1) with $T=+\infty$. If $\limsup_{t\to+\infty}\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}=+\infty$, then phenomenon is called as *infinite time gradient blow-up*, which is also referred to as *gradient grow-up*.

About this phenomenon, very few examples are known in the literature yet. In [4], Chen considered the one-dimensional case of (1.1) with slightly more general f than (A) and (B), and proved that infinite time gradient blow-up occurs for large λ. In [15], Souplet and Vázquez showed that the solution of the problem

$$\begin{cases} u_t = u_{xx} + |u_x|^p, & t>0,\ 0<x<1, \\ u(0,t)=0, \quad u(1,t)=M, & t>0, \\ u(x,0)=u_0(x), & 0<x<1, \end{cases}$$

has an infinite derivative at $x=0$ provided that $p>2$ and M equals the critical value $\frac{1}{p-2}(p-1)^{\frac{p-2}{p-1}}$. In [16], Stinner and Winkler investigated the degenerate parabolic equation

$$\begin{cases} u_t = u^p u_{xx} + k u^r u_x^2 + u^q, & t>0,\ 0<x<L, \\ u(0,0)=0=u(L,t), & t>0, \\ u(x,0)=u_0(x), & 0<x<L, \end{cases}$$

and showed that when $p>2$, $q\in[1,p-1]$, $k\geqslant 0$, and either $r=p-1$ and $k\leqslant p-2$ or $r>p-1$, under an additional assumption on u_0, the problem has a positive global solution and its gradient blows up in infinite time. In [18], Winkler further proved that the assumption on u_0 can be removed when $k=0$.

The purpose of this paper is to extend some results of Chen [4] to $n\geqslant 2$ and to give examples of infinite time gradient blow-up in higher dimensions. For cases (A), (B) and (C), we shall show that infinite time gradient blow-up happens for (1.1) under appropriate assumptions on φ and λ. Notice that for these cases, (1.1) has a unique global solution for $T=+\infty$ (see Lemma 3.1 below).

Denote by λ_1 the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Define the energy functional

$$E[u] = \int_\Omega \left(\sqrt{1+|\nabla u|^2}-1\right) - \frac{\lambda}{2}u^2\,dx. \tag{1.2}$$

Our main results are the following theorems. The first two results deal with case (A).

Theorem 1.1. *Let $f(u)=\lambda u$ and let u be the global solution of (1.1). If $E[\varphi]<0$, then infinite time gradient blow-up happens.*

Remark 1. For any given $\varphi\not\equiv 0$, by the relation

$$E[k\varphi] = \int_\Omega \left(\sqrt{1+k^2\left|\nabla\varphi(x)\right|^2}-1\right) - \frac{\lambda}{2}k^2\varphi^2(x)\,ds,$$

it is easy to see that there exists k large enough such that $E[k\varphi]<0$.

For nontrivial nonnegative initial data, we have the following results.