J. Math. Anal. Appl. 422 (2015) 544–558

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Positive solutions to some equations with homogeneous operator ☆

Fuyi Li, Chen Guan, Yuhua Li *

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, PR China

A R T I C L E I N F O

Article history:
Received 2 July 2014
Available online 29 August 2014
Submitted by J. Shi

Keywords:
Homogeneous operator
Kirchhoff equation
Positive solutions
Fixed point index

A B S T R A C T

In this paper, we discuss the positive solutions to the equation $\varphi(u)u = \lambda aAu + Bu + u_0$, where A is a positive linear completely continuous operator, B is an α-homogeneous operator defined on a cone in a real Banach space and $\varphi(u) = a + b\|u\|^\beta$. By using the fixed point index theory, when u_0 is sufficiently small, the spectral radius $\lambda r(A) < 1$ and $\alpha - \gamma\beta > 1$, where $\gamma = \operatorname{sgn} b$, we obtain a positive solution to the above equation under some appropriate conditions. The new results generalize the previous research about the homogeneous operator equation. As an application, by using our main theorem we can obtain a symmetrical positive solution to the one dimensional Kirchhoff equation.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, P be a cone in E. In this paper, we study the following equation with an α-homogeneous operator

$$\varphi(u)u = \lambda aAu + Bu + u_0, \tag{1.1}$$

where $\lambda \geqslant 0$ is a parameter, $A : E \to E$ is a positive linear completely continuous operator, $B : P \to P$ is a completely continuous α-homogeneous operator, $\varphi(u) = a + b\|u\|^\beta$ and u_0 is a given element of P.

Existence and multiplicity of solutions for nonlinear operator equations in an ordered Banach space were initiated by Amann [2]. Since then, by applying the Banach contraction mapping principle and the Krasnosel'skii fixed point theorem, many authors have studied more general nonlinear operator equations as can be seen in [3,4,9,20,21], for instance. On the other hand, the study for the Kirchhoff equation plays an important role in physics and other subjects. There have been many papers studying the Kirchhoff

☆ Partially supported by National Natural Science Foundation of China (Grant Nos. 11071149, 11301313), and Science Council of Shanxi Province (2012011004-2, 2013021001-4, 2014021009-1).

* Corresponding author. Fax: +86 0351 7011733.
E-mail address: yhli@sxu.edu.cn (Y. Li).

http://dx.doi.org/10.1016/j.jmaa.2014.08.049
0022-247X/

equations by using the variational method, see [1,5–7,12,16,17]. Recently, some papers have studied the three dimensional Kirchhoff equation by using the fixed point index theory, see [10,19]. Since we can compute the Green function for one dimensional Kirchhoff equation, we may obtain better results. Therefore, the aim of this paper is to establish the existence of positive solutions to the operator equation (1.1) and apply the results obtained to the one dimensional Kirchhoff equation so that some positive solutions are obtained by using also the fixed point index theory.

In [23], by the Krasnosel'skii fixed point theorem, the author obtained the existence of multiple positive solutions to the following equation

$$u = b(u, u) + u_0, \tag{1.2}$$

where $b : P\times P \to P$ is a bilinear and completely continuous operator with $\inf_{u\in P,\|u\|=1} \|b(u, u)\| > 0$. In [15], the authors studied the equation $u = Bu + u_0$, where B is a compact mapping with $a\|u\|^p \leqslant \|B(u)\| \leqslant b\|u\|^p$. Motivated by the results of [23] and the methods in [18], we generalize the bilinear operator b to a homogeneous operator B with $\inf_{u\in P,\|u\|=1} \|Bu\| > 0$ and add a linear operator λA. Hence, we consider the following equation with a homogeneous operator

$$u = \lambda Au + Bu + u_0, \tag{1.3}$$

where $\lambda \geqslant 0$ is a parameter. Under some appropriate conditions, we prove that Eq. (1.3) has at least a positive solution. However, in order to apply our main results to the Kirchhoff equation, we generalize further Eq. (1.3) to Eq. (1.1). By using the fixed point index theory, when u_0 is sufficiently small, the spectral radius $\lambda r(A) < 1$ and $\alpha - \gamma\beta > 1$, where $\gamma = \operatorname{sgn} b$, we obtain a positive solution to Eq. (1.1). If $\lambda = 0$, the result is one of [15]. Hence, this new abstract fixed point theorem generalizes the previous research [15,23].

In Section 3, we consider the existence of symmetrical positive solutions to the one dimensional Kirchhoff equation. By constructing a new cone in the Sobolev space $H_0^1(0, 1)$, and by using the fixed point index theory, we prove that the Kirchhoff equation has at least one, or two symmetrical positive solutions under appropriate conditions, respectively.

The key tool in our approach is the following well known lemma [8,11] originating from the work of Krasnosel'skii [13].

Lemma 1.1. *Let E be a real Banach space and $P \subset E$ be a cone in E. Let $r > 0$, and define $B_r = \{x \in E\colon \|x\| < r\}$. Assume that $A : \overline{B}_r \cap P \to P$ is a completely continuous operator such that $Ax \neq x$ for $x \in \partial B_r \cap P$.*

(i) *If $\|Au\| \leqslant \|u\|$ for $u \in \partial B_r \cap P$, then the fixed point index $i(A, B_r \cap P, P) = 1$.*
(ii) *If $\|Au\| \geqslant \|u\|$ for $u \in \partial B_r \cap P$, then $i(A, B_r \cap P, P) = 0$.*

This paper is organized as follows. In Section 2, by using the fixed point index theory we establish Theorems 2.2 and 2.5, which are our main results. In Section 3, we apply our results to the one dimensional Kirchhoff equation.

2. Positive solutions of operator equation

In this section, we first use the fixed point index theory to consider the positive solutions of Eq. (1.1).

Let E be a real Banach space with norm $\|\cdot\|$, P be a cone in E [11].

Definition 2.1. Let $B : P \to P$ be an operator. If there is $\alpha > 1$ such that