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Attractors of nonlinear dynamical systems with a weakly monotone measure

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ABSTRACT

This paper is devoted to the study of attractive sets for dynamical systems in a metric space with a measure. It is assumed that the measure of a set of points in the phase space is increasing along the flow. We prove that an invariant set is an attractor for almost all initial conditions under some extra assumptions. For a system of autonomous ordinary differential equations, we present attractivity conditions in terms of the divergence with a density function. Unlike previous results in the literature, our approach allows the use of a wider class of density functions if the divergence vanishes on a set of positive measure. As an example, the attitude stabilization problem of a rigid body is solved by using an affine feedback control for the kinematic equations in terms of quaternions.

0. Introduction

Lyapunov's direct method is known as a basic approach for the stability analysis. A new direction for the stability study of nonlinear systems was introduced by A. Rantzer. In the paper [11], he proposed a new characterization of the attractive point in terms of density functions. He proved that the existence of a density function is sufficient for the attractivity of solutions for almost all initial conditions with respect to the Lebesgue measure. This result was extended to the case when the attractor is a closed invariant set in the paper [12]. D. Angeli [1] considered concepts of almost global input-to-state stability and input-to-state stability for systems on differentiable manifolds.

In the paper [6], P. Monzon proved necessary conditions for the existence of density functions for a particular case of global asymptotical stability. S. Prajna and A. Rantzer [8] established the existence of homogeneous density functions for a homogeneous system with asymptotically stable equilibrium. In the paper [7], P. Monzon presented a result for two dimensional systems which demonstrate the relationship between the Poincaré–Bendixson theorem and density functions.

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I. Masubuchi [4] introduced a weak derivative of the measure along the trajectories of a nonlinear system. By using this notion, he proved the monotonicity condition for the integral of a density function. This concept allows to use a wider class of density functions for proving the invariance and attractivity of trajectories. S.G. Loizou and A. Jadbabaie [3] presented a method for constructing a measure by using navigation functions.

In the paper [15], J. Yu, J. Wang, and Q. Li proved that the existence of a density function with positive time derivative implies almost global stability and local asymptotical stability. Moreover, if the divergence is negative at the origin then the existence of such a density function is necessary and sufficient for almost global stability and local asymptotical stability. U. Vaidya proved a necessary and sufficient condition for almost everywhere stability of an invariant set in terms of a Lyapunov measure [14]. The concept of almost everywhere uniform stability is also studied in [10].

Our study extends Rantzer's approach for abstract dynamical systems with a monotone measure on the flow.

Another result of this paper is a theorem on sufficient conditions for the attraction of solutions of differential equations to a set, for almost all initial values, in terms of density functions. Using this theorem, we obtain conditions for the stabilizability of an equilibrium of the system describing the rotation of a rigid body in terms of quaternions.

The rest of the paper is organized as follows. Section 1 contains some auxiliary results. In Section 1, we formulate sufficient conditions for an invariant set of a dynamical systems to be the attractor for almost all initial conditions. These results are applied for studying the attractivity of solutions of ordinary differential equations in Section 2. Section 3 contains some examples that illustrate our theorems.

1. Sufficient conditions for attraction

Consider a metric space D with the distance $d : D \times D \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, +\infty)$. Suppose that there is a one-parameter family of maps

$$\varphi_t(p) \in D: \quad t \in \mathbb{R},\ p \in D, \tag{1}$$

possessing properties of a dynamical system:

1. $\varphi_0(p) = p$, for all $p \in D$;
2. $(t, p) \mapsto \varphi_t(p)$ is a continuous map from $\mathbb{R} \times D$ to D;
3. $\varphi_{t_1}(\varphi_{t_2}(p)) = \varphi_{t_1+t_2}(p)$, for all $t_1, t_2 \in \mathbb{R}$ and $p \in D$.

We use the standard notations:

$$d(x_0, M) = \inf_{y \in M} d(x_0, y), \qquad B_\varepsilon(M) = \bigl\{x \in D : d(x, M) < \varepsilon\bigr\}.$$

Let $M \subset D$, $D \setminus M \neq \emptyset$, and let μ be a measure on $D \setminus M$.

Definition 1.1. A nonempty set $M \subset D$ is an attractor of system (1) for almost all initial conditions with respect to the measure μ, if there exists a μ-measurable set $D_0 \subset D \setminus M$, $\mu((D \setminus D_0) \setminus M) = 0$, such that:

$$\lim_{t \to +\infty} d\bigl(\varphi_t(x_0), M\bigr) = 0, \quad \forall x_0 \in D_0.$$

In order to formulate the main result, we introduce a class $\mathcal{K}_0$ of continuous functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\alpha(0) = 0$.

The following theorem describes sufficient conditions for the attraction of trajectories to a set in D.