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Existence of standing waves for the complex Ginzburg–Landau equation

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ABSTRACT

We study the existence of standing wave solutions of the complex Ginzburg–Landau equation

$$\varphi_t - e^{i\theta}(\rho I - \Delta)\varphi - e^{i\gamma}|\varphi|^\alpha\varphi = 0 \tag{GL}$$

in $\mathbb{R}^N$, where $\alpha > 0$, $(N-2)\alpha < 4$, $\rho > 0$ and $\theta, \gamma \in \mathbb{R}$. We show that for any $\theta \in (-\pi/2, \pi/2)$ there exists $\varepsilon > 0$ such that (GL) has a non-trivial standing wave solution if $|\gamma - \theta| < \varepsilon$. Analogous result is obtained in a ball $\Omega \in \mathbb{R}^N$ for $\rho > -\lambda_1$, where λ_1 is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions.

1. Introduction

The complex Ginzburg–Landau equation

$$\psi_t = z_1\Delta\psi + z_2|\psi|^\alpha\psi + z_3\psi, \tag{1.1}$$

for $\alpha = 2$, $z_1, z_2, z_3 \in \mathbb{C}$, with $\Re z_1 \geq 0$ was proposed independently by Diprima, Eckhaus, Segel [8] and Stewartson, Stuart [22] to model the interaction of plane waves in fluid flows and plays a central role in the study of the development of nonlinear instabilities in fluid dynamics. See [5,24] and the references cited therein for a discussion of various problems where the complex Ginzburg–Landau equation applies. Local

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(global for $\Re z_2 < 0$) well-posedness of (1.1) (for $\alpha > 0$) was derived in both $\mathbb{R}^N$ and a domain $\Omega \subset \mathbb{R}^N$, under various boundary conditions and assumptions on the parameters, see [9,10,16,20] and the references therein.

The existence of special solutions of (1.1) (holes, fronts, pulses, sources, sinks, etc.) is discussed in numerous works, see e.g. [6,14,15,18,19,21,24]. We look for standing wave solutions. Replacing φ by $e^{i\eta t}\varphi$ for some $\eta \in \mathbb{R}$ and rescaling the equation, we rewrite (1.1) as

$$\partial_t \varphi + e^{i\theta}(\rho\varphi - \Delta\varphi) = e^{i\gamma}|\varphi|^\alpha \varphi, \tag{1.2}$$

where $\rho \in \mathbb{R}$. Given $\omega \in \mathbb{R}$, a standing wave of the form $\varphi = e^{i\omega t}u(x)$ is a solution of (1.2) if and only if u satisfies

$$i\omega u + e^{i\theta}(\rho u - \Delta u) = e^{i\gamma}|u|^\alpha u. \tag{1.3}$$

Plane waves $\varphi = e^{i(kx-\omega t)}$, where $k, \omega \in \mathbb{R}$ are particular standing waves. It is easy to see that (1.2) admits plane wave solutions in $\mathbb{R}^N$ for all values of ρ, θ, γ and α. Stationary solutions are also standing waves of special kind. In the case of the nonlinear heat equation $\theta = \gamma = 0$ or $\theta = 0$, $\gamma = \pi$ then $\omega = 0$, so that Eq. (1.3) reduces to the nonlinear elliptic equation $\rho u - \Delta u = \pm|u|^\alpha u$. The case of the nonlinear Schrödinger equation $\theta = \pm\gamma = \pm\frac{\pi}{2}$ leads to the equation $(\rho \pm \omega)u - \Delta u = \pm|u|^\alpha u$.

We obtain here solutions that are different from these particular ones. In fact, using well known results of the theory of nonlinear elliptic equations for the case $\omega = 0$ and $\theta = \gamma$, we show the existence of nontrivial standing wave solutions for $\theta \approx \gamma$ by a perturbation argument, as we describe below.

Eq. (1.3) will be considered both in the whole space $\Omega = \mathbb{R}^N$ or in a ball $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition, for $N \geq 1$. We suppose $\theta, \gamma \in (-\pi/2, \pi/2)$ and α subcritical, i.e.

$$0 < \alpha, \qquad (N-2)\alpha < 4, \tag{1.4}$$

which includes the relevant case $\alpha = 2$, for $N \leq 3$. For $\theta = \gamma$ and $\omega = 0$, (1.3) reduces to

$$\rho u - \Delta u - |u|^\alpha u = 0. \tag{1.5}$$

Consider first $\Omega = \mathbb{R}^N$, in which case we assume that $\rho > 0$. It was shown in [13] that (1.5) has a unique positive radially symmetric solution $U \in C^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. ($C_0(\mathbb{R}^N)$ is the space of continuous functions which tend to zero at infinity.) In fact, $U \in H^2_{\mathrm{rad}}(\mathbb{R}^N)$, the subspace of radial functions of $H^2(\mathbb{R}^N)$. Note that (1.5) is phase invariant, i.e., $Ue^{i\beta} \in \mathbf{H}^2_{\mathrm{rad}}(\mathbb{R}^N)$ is also a solution for all $\beta \in \mathbb{R}$. Here and in the rest of this paper we consider real spaces composed of complex-valued functions, and distinguish them from real spaces of real-valued functions by using bold face typing.

Theorem 1.1. *Assume (1.4) holds and suppose $\rho > 0$. Let $U \in H^2_{\mathrm{rad}}(\mathbb{R}^N)$ be the unique positive radial solution of (1.5). Given $\theta \in (-\pi/2, \pi/2)$ and $\beta \in \mathbb{R}$ there exists $0 < \varepsilon < \min\{\pi/2 - \theta, \pi/2 + \theta\}$ and a C^1 mapping $g : (\theta - \varepsilon, \theta + \varepsilon) \to \mathbb{R} \times \mathbf{H}^2_{\mathrm{rad}}(\mathbb{R}^N)$, $g(\gamma) = (\omega_\gamma, u_\gamma)$, satisfying $\omega_\theta = 0$, $u_\theta = Ue^{i\beta}$ and such that $\varphi_\gamma = e^{i\omega_\gamma t}u_\gamma$ is a solution of (1.2).*

In the bounded domain case of the unitary ball Ω of $\mathbb{R}^N$, we suppose that

$$\rho > -\lambda_1, \tag{1.6}$$

where λ_1 is the first eigenvalue associate to the Laplace–Dirichlet operator in Ω. As in the case of the whole space, (1.5) admits a unique positive solution $U \in H^2(\Omega) \cap H^1_0(\Omega)$, which is radial and radially decreasing. The following result is analogous to Theorem 1.1.