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The exit time of planar Brownian motion and the Phragmén–Lindelöf principle

Greg Markowsky

Monash University, Victoria 3800, Australia

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ABSTRACT

In this paper a version of the Phragmén–Lindelöf principle is proved using probabilistic techniques. In particular, we will show that if the pth moment of the exit time of Brownian motion from a planar domain is finite, then an analytic function on that domain is either bounded by its supremum on the boundary or else goes to ∞ along some sequence more rapidly than $e^{|z|^{2p}}$. We also provide a method of constructing domains whose exit time has finite pth moment. This allows us to give a general Phragmén–Lindelöf principle for spiral-like and star-like domains, as well as a new proof of a theorem of Hansen. A number of auxiliary results are presented as well.

1. Introduction

The Phragmén–Lindelöf principle is a method by which the maximum modulus principle can be generalized to certain unbounded domains in $\mathbb{C}$. The principle roughly states that, on particular domains, analytic functions must either be bounded by their supremum on the boundary of the domain or tend rapidly to ∞ along some sequence. We note that, as the principle is generally stated, the precise meaning of "tend rapidly to infinity" will depend upon the domain in question. Our aim in this paper is to prove a general form of the principle using probabilistic arguments, in particular a relationship between the growth of functions and the moments of exit times of planar Brownian motion. We will also show how the principle can be applied in a number of special cases.

In order to give a precise statement of our main result, we need a few definitions. In what follows, B_t will always refer to a planar Brownian motion. For any domain $W \subseteq \mathbb{C}$ we let $T_W = \inf\{t \geq 0 : B_t \notin W\}$ be the first exit time of Brownian motion from W. The notations E_a and P_a will be used to refer to expectation and probability conditioned upon $B_0 = a$ a.s. If $E_a[T_W^p] < \infty$ for some $a \in W$, $p > 0$, then the connectedness of W implies that $E_b[T_W^p] < \infty$ for all $b \in W$ (see [2, (3.13)]), and we will in this case simply

E-mail address: gmarkowsky@gmail.com.

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write $E[T_W^p] < \infty$. δW denotes the boundary of W in $\mathbb{C}$; that is, δW does not include the point at ∞. We will prove the following theorem.

Theorem 1. *Let W be a domain such that $E[T_W^p] < \infty$. Suppose that f is an analytic function on W such that $\limsup_{z \longrightarrow \delta W} |f(z)| \le K < \infty$, and $|f(z)| \le Ce^{C|z|^{2p}} + C$ for some $C > 0$. Then $|f(z)| \le K$ for all $z \in W$.*

The proof will be given in Section 2. The theorem encompasses some well-known special cases, as well as some which appear to be new, as will be shown in Section 3. For instance, formulations are available for an infinite wedge and arbitrary simply connected domains, as well as for general star-like and spiral-like domains. Further formulations are possible which make use of a method, presented in Section 4, of building domains whose exit time has finite pth moment.

2. Proof of Theorem 1

The key to our investigation will be the following pair of results, which we will collectively refer to as *Burkholder's Theorem*. Following [2], we define $B_\tau^* = \sup_{0 \le t < \infty} |B_{t \wedge \tau}|$.

Theorem *(Burkholder). (i) For any $p \in (0,\infty)$ there are constants $c_p, C_p > 0$ such that for any stopping time τ we have*

$$c_p E_a\big[\big(\tau + |a|^2\big)^p\big] \le E_a\big[\big(B_\tau^*\big)^{2p}\big] \le C_p E_a\big[\big(\tau + |a|^2\big)^p\big]. \tag{2.1}$$

In particular, $E_a[\tau^p] < \infty$ if, and only if, $E_a[(B_\tau^*)^{2p}] < \infty$.

(ii) For any $p \in (0,\infty)$ there is a constant $C_p > 0$ such that for any stopping time τ with $E_a[\ln \tau] < \infty$ we have

$$E_a\big[|B_\tau|^{2p}\big] \le E_a\big[\big(B_\tau^*\big)^{2p}\big] \le C_p E_a\big[|B_\tau|^{2p}\big]. \tag{2.2}$$

It may be tempting to see part (i) at least as a straightforward consequence of the standard Burkholder–Davis–Gundy inequality by separately bounding the supremums of the real and imaginary parts of B_t; however, this argument is not quite valid, since a stopping time for B_t need not be a stopping time for its projection onto the real or imaginary axis. The reader who would like to see a proof of the theorem is therefore referred to [2].

Before proving Theorem 1, we give a preliminary result on subharmonic functions. In what follows, $cl(W)$ will denote the closure of the set W in $\mathbb{C}$.

Proposition 1. *Let W be a domain with $E[T_W^p] < \infty$. Suppose that u is a continuous function on $cl(W)$ which is subharmonic on W and satisfies $\sup_{z \in \delta W} u(z) \le K$, for some $K > 0$. Suppose further that $u(z) \le C|z|^{2p} + C$ for some $C < \infty$. Then $u(z) \le K$ for all $z \in W$.*

Proof. Let $S_M = \inf\{t \ge 0 : |B_t| = M\}$, and fix $a \in W$. Since u is subharmonic, $u(a) \le E_a[u(B_{T_W \wedge S_M})]$ (see [4, Secs. 2.IV.3 and 2.IX.3]). We would like to let $M \longrightarrow \infty$ to obtain $u(a) \le E_a[u(B_{T_W})] \le K$; note that the conditions on u imply that $|u(B_{T_W \wedge S_M})| \le C|B_{T_W \wedge S_M}|^{2p} + C \le C(B_{T_W}^*)^{2p} + C$, and $E_a[C(B_{T_W}^*)^{2p} + C] < \infty$ by Burkholder's Theorem. The dominated convergence theorem therefore applies, and we get $u(a) \le \lim_{M \nearrow \infty} E_a[u(B_{T_W \wedge S_M})] = E_a[u(B_{T_W})] \le K$. □

Proof of Theorem 1. We may assume $K = 1$. Set

$$\log^+ x = \begin{cases} \log x & x > 1, \\ 0 & x \le 1. \end{cases} \tag{2.3}$$