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Chebyshev centers and fixed point theorems

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ABSTRACT

Brodskii and Milman proved that there is a point in $C(K)$, the set of all Chebyshev centers of K, which is fixed by every surjective isometry from K into K whenever K is a nonempty weakly compact convex subset having normal structure in a Banach space. Motivated by this result, Lim et al. raised the following question namely "does there exist a point in $C(K)$ which is fixed by every isometry from K into K?". In fact, Lim et al. proved that "if K is a nonempty weakly compact convex subset of a uniformly convex Banach space, then the Chebyshev center of K is fixed by every isometry T from K into K". In this paper, we prove that if K is a nonempty weakly compact convex set having normal structure in a strictly convex Banach space and $\mathfrak{F}$ is a commuting family of isometry mappings on K then there exists a point in $C(K)$ which is fixed by every mapping in $\mathfrak{F}$.

1. Introduction

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive (isometry) if $\|Tx - Ty\| \leq \|x - y\|$ ($\|Tx - Ty\| = \|x - y\|$) for $x, y \in K$.

Theorem 1.1. *(See [7].) If K is a nonempty bounded closed convex subset of a uniformly convex Banach space and $T : K \to K$ is a nonexpansive mapping, then T has a fixed point in K.*

The above theorem was proved independently by F. Browder and D. Göhde. At the same time Kirk established a more general result:

Theorem 1.2. *(See [8].) If K is a nonempty weakly compact convex set having normal structure in a Banach space X and T is a nonexpansive mapping from K into K, then T has a fixed point in K.*

A well known result regarding common fixed points of a family of nonexpansive mappings by Belluce and Kirk is:

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Theorem 1.3. *(See [1].) Let K be a nonempty weakly compact convex set in a strictly convex Banach space and $\mathfrak{F}$ be a nonempty commuting family of nonexpansive mappings from K into K. Assume K has normal structure, then there exists $x \in K$ such that $Tx = x$ for every $T \in \mathfrak{F}$.*

The above Theorem 1.3 was further generalized to Banach spaces by Lim in [9].

Theorem 1.4. *(See [9].) Let K be a nonempty weakly compact convex set in a Banach space and $\mathfrak{F}$ be a nonempty commuting family of nonexpansive mappings from K into K. Assume K has normal structure, then there exists $x \in K$ such that $Tx = x$ for every $T \in \mathfrak{F}$.*

In the proof of Theorem 1.4, a concept called asymptotic center of a net was applied. This concept, asymptotic center, was introduced by Edelstein in [6] for a bounded sequence in a convex subset C of a uniformly convex Banach space X.

The notion of asymptotic center was extended to a decreasing net of bounded subsets of a Banach space by Lim in [12] and it was also shown that:

Lemma 1.1. *(See [12,13].) Let K be a nonempty weakly compact convex set in a Banach space and $T : K \to K$ be a nonexpansive map. Then the set of all asymptotic centers of $\{T^n(K) : n = 0, 1, 2, ...\}$ is a T invariant set.*

For other applications of the concept asymptotic center to fixed point theory one may refer to [9–13].

Notice that the geometric property, normal structure, was used in Theorem 1.2, Theorem 1.3 and Theorem 1.4. The notion of normal structure, introduced by Brodskii and Milman in [3], is defined as follows:

Definition 1.1. (See [3,7].) A nonempty bounded convex set K in a Banach space X is said to have normal structure if for every nonempty convex set $C \subseteq K$ with more than one point has a point $x \in C$ such that $r(x, C) < \delta(C)$, where $r(x, C) = \sup\{\|x - y\| : y \in C\}$ and $\delta(C) = \sup\{\|z - y\| : z, y \in C\}$.

Define $r(K) = \inf\{r(x, K) : x \in K\}$ and $C(K) = \{x \in K : r(x, K) = r(K)\}$. Then the set $C(K)$ and the number $r(K)$ are called, respectively, the set of Chebyshev centers of K and the Chebyshev radius of K.

It is also known that [3] Brodskii and Milman proved the following interesting result:

Theorem 1.5. *(See [3].) Let K be a nonempty weakly compact convex subset of a Banach space X and $\mathfrak{F} = \{T : K \to K : T$ is a surjective isometry mapping$\}$. Assume K has normal structure, then there exists $x \in C(K)$ such that $Tx = x$ for every $T \in \mathfrak{F}$.*

It is known that [13] if T is an isometry from K onto K then $C(K)$ is invariant under T. In fact, it is apparent that $T(C(K)) = C(K)$.

Motivated by the results of Brodskii and Milman and the fact $T(C(K)) = C(K)$ whenever T is a surjective isometry on K, Lim et al. raised the following questions in [13]:

Question 1. Let T be an isometry on K which is not surjective. Does one still have $T(C(K)) \subseteq C(K)$?

Question 2. Let K be a weakly compact convex subset of a Banach space and assume K has normal structure. Does there exist a point in $C(K)$ which is fixed by every isometry from K into K?

In case of uniformly convex Banach spaces, Lim et al. affirmatively answered the above questions. In fact, the authors in [13] proved: