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# Pointwise versus equal (quasi-normal) convergence via ideals

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ABSTRACT

We prove a characterization showing when the ideal pointwise convergence does not imply the ideal equal (aka quasi-normal) convergence. The characterization is expressed in terms of a cardinal coefficient related to the bounding number $\mathfrak{b}$. We also prove a characterization showing when the ideal equal limit is unique.



## 1. Introduction

By an *ideal on a set* $X$ we mean a nonempty family of subsets of $X$ closed under taking finite unions and subsets of its elements. By $\mathrm{Fin}(X)$ we denote the ideal of all finite subsets of $X$ (for $X = \mathbb{N}$ we write $\mathrm{Fin} = \mathrm{Fin}(\mathbb{N})$). Moreover, we also assume that ideals are *proper* (i.e. $X \notin \mathcal{I}$) and contain all finite subsets of $X$ (i.e. $\mathrm{Fin}(X) \subseteq \mathcal{I}$).

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. We say that a sequence $(x_n)$ of reals is $\mathcal{I}$-*convergent* to $x \in \mathbb{R}$ if the set $\{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. We write $(x_n) \xrightarrow{\mathcal{I}} x$ in this case. For $\mathcal{I} = \mathrm{Fin}$ we obtain the ordinary convergence.

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on $\mathbb{N}$. Let $f_n$ $(n \in \mathbb{N})$ and $f$ be real-valued functions defined on a set $X$. We say that the sequence $(f_n)$ is $(\mathcal{I}, \mathcal{J})$-*equally convergent* to $f$ if there exists a sequence of positive reals $(\varepsilon_n) \xrightarrow{\mathcal{J}} 0$ such that $\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$ [13]. We write $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ in this case. For $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$ we obtain the equal convergence introduced by Á. Császár and M. Laczkovich [9] and write $(f_n) \xrightarrow{e} f$ instead of $(f_n) \xrightarrow{(\mathrm{Fin},\mathrm{Fin})-e} f$. The equal convergence was independently introduced by

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Z. Bukovská (see [2] and [1]), where she used the name "quasi-normal convergence" instead of "equal convergence". In [4,3] the authors use the equal convergence in the theory of trigonometrical series. In [11] and [14] the authors considered the ideal equal convergence in some special cases. Namely, in [11] the authors considered $(\mathcal{I},\mathcal{I})$-equal convergence, whereas in [14] the authors considered $(\mathcal{I},\mathrm{Fin})$-equal convergence.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let $f_n$ $(n\in\mathbb{N})$ and $f$ be real-valued functions defined on a set $X$. We say that the sequence $(f_n)$ is *$\mathcal{I}$-pointwise convergent* to $f$ if $\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}$ for every $x \in X$ and $\varepsilon > 0$. We write $(f_n) \xrightarrow{\mathcal{I}} f$ in this case. For $\mathcal{I} = \mathrm{Fin}$ we obtain the pointwise convergence and write $(f_n) \to f$.

If $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$, then it is not difficult to show that the equal convergence implies the pointwise convergence. However the same is not true for arbitrary ideals $\mathcal{I}$, $\mathcal{J}$.

**Theorem 1.1.** *(See [13, Proposition 4.4].) Let $\mathcal{I}$, $\mathcal{J}$ be ideals on $\mathbb{N}$. The following are equivalent.*

(1) *For every sequence of real-valued functions defined on $X$, if $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$ then $(f_n) \xrightarrow{\mathcal{I}} f$.*
(2) *$\mathcal{J} \subseteq \mathcal{I}$.*

If $\mathcal{I} = \mathcal{J} = \mathrm{Fin}$, then it is also not difficult to show that the pointwise convergence does not imply the equal convergence. In [11, Example 3.1] the authors showed that if $\mathcal{I}$ is a countably generated ideal, then the $\mathcal{I}$-pointwise convergence does not imply the $(\mathcal{I},\mathcal{I})$-equal convergence, and in [13, Example 4.7] we showed that if $|X| \geq \mathfrak{c}$ and $\mathcal{J} \subseteq \mathcal{I}$ then the $\mathcal{I}$-pointwise convergence does not imply the $(\mathcal{I},\mathcal{J})$-equal convergence. (The relationship between pointwise convergence and equal convergence in the realm of continuous functions was already done in the literature, for instance in [7] and [8] the authors consider the case of the ordinary convergence (i.e. $\mathcal{I} = \mathrm{Fin}$), whereas in [10] and [6] the authors generalized the previous results to the case of the ideal convergence.)

The aim of this paper is to prove the following theorem that gives a sufficient and necessary condition when the $\mathcal{I}$-pointwise convergence implies the $(\mathcal{I},\mathcal{J})$-equal convergence (see Section 5 for the proof).

**Theorem 1.2.** *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on $\mathbb{N}$. For every set $X$ the following are equivalent.*

(1) *For every sequence $(f_n)$ of real-valued functions defined on $X$, if $(f_n) \xrightarrow{\mathcal{I}} f$ then $(f_n) \xrightarrow{(\mathcal{I},\mathcal{J})-e} f$.*
(2) *For every family $\{E_n^\alpha : n \in \mathbb{N}, \alpha < |X|\} \subseteq \mathcal{I}$ such that $E_n^\alpha \cap E_k^\alpha = \emptyset$ for $n \neq k$, $\alpha < |X|$ there exists a partition $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{J}$ of $\mathbb{N}$ such that*

$$\bigcup_{n\in\mathbb{N}} \left( A_n \cap \bigcup_{i\leq n} E_i^\alpha \right) \in \mathcal{I}$$

*for every $\alpha < |X|$.*

In Sections 2, 3 and 4 we introduced tools we need to prove the above-mentioned characterization. For instance we define a cardinal coefficient $\mathfrak{b}(\mathcal{I},\mathcal{J})$ which is related to the bounding number $\mathfrak{b}$. In Section 6 we prove a characterization showing when the $(\mathcal{I},\mathcal{J})$-equal limit is unique.

## 2. Some properties of ideals

### 2.1. Orthogonal ideals

We say that ideals $\mathcal{I}$, $\mathcal{J}$ on $X$ are *orthogonal* if there is a set $A \subseteq X$ such that $A \in \mathcal{I}$ and $X \setminus A \in \mathcal{J}$.