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Global existence and nonexistence of solution for Cauchy problem of two-dimensional generalized Boussinesq equations

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A R T I C L E   I N F O

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A B S T R A C T

In this paper we consider the Cauchy problem of two-dimensional generalized Boussinesq-type equation $u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + \Delta f(u) = 0$. Under the assumption that $f(u)$ is a function with exponential growth at infinity and under some assumptions on the initial data, we prove the existence and nonexistence of global weak solution. There are very few works on Boussinesq equation with nonlinear exponential growth term by potential well theory.

1. Introduction

We are concerned with the following Cauchy problem:

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u + \Delta f(u) = 0, \quad x \in R^2,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in R^2, \tag{1.2}$$

where $u(x,t)$ denotes the unknown function, $f(s)$ is the given nonlinear function with exponential growth like e^{s^2} at infinity, $u_0(x)$ and $u_1(x)$ are given initial value functions, the subscript t indicates the partial derivative with respect to t, and Δ denotes the Laplace operator in R^2.

This model arises in a number of mathematical models of physical processes, for example in the modeling of surface waves in shallow waters or considering the possibility of energy exchange through lateral surfaces of the waveguide in the physical study of nonlinear wave propagation in waveguide [7,25,24]. Due to the mathematical and physical importance, over the last couple of decades, global existence and nonexistence of solutions of Boussinesq type equations have been extensively studied by many mathematicians and

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physicists. In the one-dimensional case, a great deal of effort has been made to establish the sufficient condition for the existence or nonexistence of global solution to various nonlinear terms, such as power-type nonlinearities $f(s) = \pm a|s|^p$, $f(s) = a|s|^p s$, $p > 1$ (see, for example, [3,28,5,15–18] and the references therein), or more general nonlinearities satisfying either sign properties or some additional structure growth conditions such as $f'(s) \geq C$ (bounded below) or $f(s) = \sum_{k=1}^{m} a_k |s|^p s$ (see [6,31,13,34] and the references therein).

To the authors' best knowledge, the multi-dimensional version of Boussinesq equation is less studied. Most recently, in [29,30,21,22,36,37,35,38,40,32,11,12], the authors considered the Cauchy problem of the multi-dimensional nonlinear evolution equation for constant k:

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - k\Delta u_t = \Delta f(u), \quad x \in R^n,\ t > 0. \tag{1.3}$$

They gave the existence of local and global solution and the nonexistence of global solution. Note that in [29,30,21,22,36,37,35,38,40,32,11,12] in order to obtain the global existence of solution for problem (1.3) the authors requested that $f(s)$ possesses polynomial growth. In [21,22], Polat and Ertas studied the existence, both locally and globally in time, asymptotic and blow-up of a solution for the problem (1.3) under the assumptions $f'(s) \geq C$ and $|f'(u)| \leq a|u|^p + b$. In [37] the authors requested $F(s) \geq 0$ or $f'(s) \geq C$ (bounded below), so the global existence of solution for Cauchy problem of (1.3) was solved. We point that the conditions of Lemma 2.2 and Corollary 2.3 in [37] are essentially power-type nonlinearity, i.e. $|f(u)| \leq a|u|^p + b|u|^q$, where a, b are positive constants, although they requested $F(s) \geq 0$ or $f'(s) \geq C$ (bounded below).

However, the results of [29,30,21,22,36,37,35,38,40,32,11,12] are not applicable for (1.1) with exponential growth like e^{s^2} at infinity. Unlike other studies, we focus on the nonlinear Boussinesq type equations with exponential nonlinearities such as e^{s^2} at infinity. As far as we are concerned, this is the first work in the literature that takes into account the exponential growth of the function f for multi-dimensional Boussinesq equation. In this paper we consider the existence and nonexistence for Cauchy problem (1.1)–(1.2) for two-dimensional case. The motivation of studying problem (1.1)–(1.2) with nonlinearities of exponential growth comes from the following nonlinear damped wave equation

$$u_{tt} - \Delta u + h(u_t) = g(u), \quad \text{in } \Omega \times (0, +\infty), \tag{1.4}$$

where Ω is a bounded domain of R^2. In the recent works [19,1], the authors showed the global existence as well as blow-up of solutions for the problem (1.4) with exponential nonlinear term g, by taking the initial data inside the potential well [26,10]. Moreover, they also got the optimal and uniform decay rates of the energy for global solutions. The ingredients used in their proof [19,1] are essentially the Trudinger–Moser inequality (see [20,27]) and the well-known mountain pass level due to Ambrosetti and Rabinowitz [2].

Recently, we have proved the existence of global weak solution of the Cauchy problem for the following equation

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u = \Delta f(u), \tag{1.5}$$

if $f(u)$ is the nonlinear function with exponential growth like e^{s^2} at infinity [9]. In this case, it is easy to get the result by Galerkin's method because exponential nonlinear term $f(u)$ is on the right of Eq. (1.5). In this paper, we consider problem (1.1)–(1.2), where exponential nonlinear term $f(u)$ is on the left of Eq. (1.1), by potential well combined with Galerkin's method. The main purpose of this article is to establish the sufficient conditions for existence and nonexistence of global solution to the Cauchy problem (1.1)–(1.2) with an exponential growth nonlinear term for spaces–dimensions $n = 2$, by considering similar arguments due to [19,1]. The ingredients used in our proof are essentially the Trudinger–Moser inequality when $\Omega = R^2$ (see [4,8]) and Galerkin's methods.