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# Asymptotic expansions of the moments of extremes from general error distribution

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ABSTRACT

In this paper, we derive the asymptotic expansions of the moments of normalized partial maxima for general error distribution. A byproduct is to deduce the convergence rates of the moments of normalized maxima to the moments of the corresponding extreme value distribution.



## 1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with marginal cumulative distribution function (cdf) $F_v \sim \mathrm{GED}(v)$, the general error distribution with shape parameter $v > 0$. The probability density function (pdf) of $\mathrm{GED}(v)$ is defined by

$$f_v(x) = \frac{v \exp(-(1/2)|x/\lambda|^v)}{\lambda 2^{1+1/v} \Gamma(1/v)}, \quad x \in \mathbb{R},$$

where $\lambda = [2^{-2/v}\Gamma(1/v)/\Gamma(3/v)]^{1/2}$ and $\Gamma(\cdot)$ denotes the gamma function [10]. For $v = 2$, $\mathrm{GED}(2)$ reduces to the standard normal distribution.

Recently, asymptotic behaviors related to $\mathrm{GED}(v)$ have been studied in the literature. Peng et al. [11] established the Mills ratio and distributional tail representation of $\mathrm{GED}(v)$, and showed that $F_v \in D(\Lambda)$, i.e., there exist norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\mathbb{P}(M_n \leq a_n x + b_n) - \Lambda(x)\right| = 0, \tag{1.1}$$

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where $M_n = \max_{1\le k\le n} X_k$ denotes the partial maximum of $\{X_n,\ n \ge 1\}$ and $\Lambda(x) = \exp(-e^{-x})$, the Gumble extreme value distribution. Uniform convergence rate of $\mathbb{P}(M_n \le a_n x + b_n) - \Lambda(x)$ has been established by Peng et al. [12] which extended the work of Hall [1] for the case of GED(2). For higher-order expansion of $\mathbb{P}(M_n \le a_n x + b_n)$, see Nair [9] for GED(2) and Jia and Li [3] for general GED($v$) with shape parameter $v > 0$.

Moments convergence of extremes was studied by McCord [8], Pickands [13] and Ramachandran [14], see Section 2.1 in Resnick [15]. The relationship between weak convergence and moment convergence of order statistics was considered by Hill and Spruill [2]. The objective of this paper is to establish the higher-order expansions of the moments of $M_n$ for GED($v$). Nair [9] derived the higher-order expansions of moments of extremes for standard normal distribution GED(2). Recall that Peng et al. [12] showed that (1.1) holds with norming constants $a_n$ and $b_n$ satisfying the following equations:

$$1 - F_v(b_n) = n^{-1} \quad \text{and} \quad a_n = 2v^{-1}\lambda^v b_n^{1-v}. \tag{1.2}$$

Noting that for $v = 1$ the norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log n/2$. By Proposition 2.1(iii) in Resnick [15], we have

$$\Delta_r(n) = \mathbf{E}\left(\frac{M_n - b_n}{a_n}\right)^r - \int_{x\in\mathbb{R}} x^r\, d\Lambda(x) \to 0 \tag{1.3}$$

as $n \to \infty$ for all nonnegative integers $r$. The following work is to establish the asymptotic expansions of $\mathbf{E}((M_n - b_n)/a_n)^r$, from which we can derive the convergence rates of $\Delta_r(n)$. For more related work on asymptotic expansions of distributions and moments of extremes for given distributions, see Peng et al. [12], Liao and Peng [4] and Liao et al. [5–7].

This paper is organized as follows. Section 2 provides the main results, and some necessary auxiliary lemmas and their proofs are given in Section 3. The proofs of the main results are given in Section 4.

## 2. Main results

In this section, we provide the main results. In the sequel, for nonnegative integers $r$ let

$$m_r(n) = \mathbf{E}\left(\frac{M_n - b_n}{a_n}\right)^r = \int_{x\in\mathbb{R}} x^r dF_v^n(a_n x + b_n)$$

and

$$m_r = \mathbf{E}\xi^r = \int_{x\in\mathbb{R}} x^r d\Lambda(x)$$

respectively denote the $r$th moments of $(M_n - b_n)/a_n$ and $\xi \sim \Lambda(x)$, and the norming constants $a_n$ and $b_n$ are given by (1.2). The main results are stated as follows.

**Theorem 2.1.** *Let $\{X_n,\ n \ge 1\}$ be an iid sequence with marginal distribution $F_v \sim \mathrm{GED}(v)$. Then,*

(i) *for $v \ne 1$, with norming constants $a_n$ and $b_n$ given by (1.2), we have*

$$b_n^v\left[b_n^v\left(m_r(n) - m_r\right) + \left(1 - v^{-1}\right)\lambda^v r\left(m_{r+1} + 2m_r\right)\right]$$