J. Math. Anal. Appl. 422 (2015) 1213–1234

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Nonlinear stability of traveling wave solutions for the compressible fluid models of Korteweg type

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ARTICLE INFO

Article history:
Received 9 May 2014
Available online 28 September 2014
Submitted by K. Nishihara

Keywords:
Navier–Stokes–Korteweg system
Traveling wave solutions
Nonlinear stability
L^2-energy estimates

ABSTRACT

This paper is concerned with the existence and time-asymptotic nonlinear stability of traveling wave solutions to the Cauchy problem of the one-dimensional compressible fluid models of Korteweg type, which governs the motions of the compressible fluids with internal capillarity. The existence of traveling wave solutions is obtained by the phase plane analysis, then the traveling wave solution is shown to be asymptotically stable by the elementary L^2-energy method.

1. Introduction

In this paper, we consider the existence and time-asymptotic nonlinear stability of traveling wave solutions to the Cauchy problem of the one-dimensional compressible fluid models of Korteweg type in the Lagrangian coordinates

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu\left(\dfrac{u_x}{v}\right)_x + \kappa\dfrac{1}{v}\left(\dfrac{1}{v}\left(\dfrac{1}{v}\left(\dfrac{1}{v}\right)_x\right)_x\right)_x, \end{cases} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$

with initial data

$$\big(v(t,x), u(t,x)\big)\big|_{t=0} = \big(v_0(x), u_0(x)\big) \to (v_\pm, u_\pm), \quad \text{as } x \to \pm\infty. \tag{1.2}$$

Here the unknown functions are the specific volume $v > 0$, the velocity u, and the pressure $p = p(v)$ of the fluids respectively. $\mu > 0$ and $\kappa > 0$ are the viscosity coefficient and capillary coefficient, respectively, and

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http://dx.doi.org/10.1016/j.jmaa.2014.09.050
0022-247X/

$v_\pm > 0$, $u_\pm$ are given constants. Throughout this paper, we assume that μ and κ are positive constants and the pressure $p(v)$ is a positive smooth function for $v > 0$ satisfying

$$p'(v) < 0, \qquad p''(v) > 0, \quad \forall v > 0. \tag{1.3}$$

System (1.1) can be used to describe the motion of the compressible isothermal viscous fluids with internal capillarity. Notice that when $\kappa = 0$, it is reduced to the compressible Navier–Stokes system. The formulation of the theory of capillarity with diffuse interface was first introduced by Korteweg [20], and derived rigorously by Dunn and Serrin [8]. The interested readers are referred to [1,3,10] for further information on the Korteweg type model.

There have been many mathematical results on the compressible Navier–Stokes–Korteweg system. We refer to [2,6,7,11–14,21,33,35,23,29,30] and the references therein. However, most of these results are concentrated on the case when the far-fields of the initial data are equal and consequently the large time behaviors of its global solutions are described by the constant states. To the best of our knowledge, fewer results have been obtained for the case when the far-fields of the initial data are different. In this case, the large time behaviors of solutions can be described by some nonlinear elementary waves, such as rarefaction waves, viscous shock profiles, viscous contact wave, etc. Recently, we [4,5] consider the large time behavior of solutions to the one-dimensional isothermal compressible fluid models of Korteweg type toward rarefaction waves and viscous contact wave. As a continuation of [4,5], we study in this paper the time-asymptotic behavior of solutions of the Korteweg-type models toward the viscous shock profile. Notice that the existence of traveling wave solutions for the one-dimensional compressible fluid models of Korteweg type has been studied by Slemrod in [29,30], where the Korteweg tensor is given by $-\mu^2 A v_{xxx}$ with A and the viscosity coefficient $\mu > 0$ being constant. However, the time-asymptotic nonlinear stability of the traveling wave solutions for the compressible fluid models of Korteweg type is still unknown up to now. The main purpose of this article is to establish such a result.

Now we begin to formulate our main results. It is well known that for $v > 0$, the condition (1.3) guarantees the corresponding Euler system of (1.1) (i.e. (1.1) with $\mu = \kappa = 0$) to be strictly hyperbolic with two distinct eigenvalues

$$\lambda_1(v) = -\sqrt{-p'(v)} < \sqrt{-p'(v)} = \lambda_2(v), \tag{1.4}$$

in which both eigenvalues are genuinely nonlinear.

We look for the traveling wave solutions to system (1.1) of the form

$$(v,u)(x,t) = (V,U)(\xi), \quad \xi = x - st, \tag{1.5}$$

where s denotes the traveling wave speed and ξ is the traveling wave variable. For convenience of study, we rewrite the system (1.1) in the conservative form:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu\left(\dfrac{u_x}{v}\right)_x + \kappa\left(\dfrac{-v_{xx}}{v^5} + \dfrac{5v_x^2}{2v^6}\right)_x, \end{cases} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{1.6}$$

Substituting (1.5) into (1.6), we obtain the equations for $(V,U)(\xi)$:

$$\begin{cases} -sV_\xi - U_\xi = 0, \\ -sU_\xi + p(V)_\xi = \mu\left(\dfrac{U_\xi}{V}\right)_\xi + \kappa\left(\dfrac{-V_{\xi\xi}}{V^5} + \dfrac{5V_\xi^2}{2V^6}\right)_\xi \end{cases} \tag{1.7}$$

with the boundary conditions