Growth at infinity and index of polynomial maps✩

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ARTICLE INFO

Article history:
Received 11 May 2014
Available online 30 September 2014
Submitted by A. Daniilidis

Keywords:
Topological index
Polynomial vector fields
Newton polyhedron
Łojasiewicz exponent at infinity

ABSTRACT

Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map such that $F^{-1}(0)$ is compact, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then we give a condition implying that there is a uniform bound for the Łojasiewicz exponent at infinity in certain deformations of $F$. This fact gives a result about the invariance of the global index of $F$.

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1. Introduction

Given a polynomial map $F: \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{-1}(0)$ is finite, in this article we study the problem of determining which monomials can be added to each component function of $F$ leading to a map having the same global index than $F$. We recall that the global index of $F$, which we denote by $\text{ind}(F)$, is defined as $\text{ind}(F) = \sum_{x \in F^{-1}(0)} \text{ind}_x(F)$, where $\text{ind}_x(F)$ denotes the index, or topological degree, of $F$ at each point $x \in F^{-1}(0)$. The local version of this question, which is analyzed in the articles [1,8,15,20], takes part in the wider problem of determining which monomials in the Taylor expansion of a smooth vector field determine the local phase portrait (see for instance [3], [4] and [28]). The first step in this approach to the study of global indices is the result of Cima, Gasull and Mañosas [7, Proposition 2] on the index of maps whose monomials of maximum degree with respect to some vector of weights have an isolated zero. We call these maps pre-weighted homogeneous (see Definition 7.1 for a precise formulation of this concept).

Apart from [7], our motivation to study global indices comes from the estimation of the Łojasiewicz exponent at infinity of a given polynomial map $F: \mathbb{K}^n \to \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (see the article of Krasiński [17] for a detailed survey about Łojasiewicz exponents at infinity). This number, which is denoted
by \( \mathcal{L}_\infty(F) \), is defined as the supremum of those real numbers \( \alpha \) such that there exist constants \( C, M > 0 \) such that

\[
\|x\|^\alpha \leq C\|F(x)\|
\]

for all \( x \in \mathbb{K}^n \) such that \( \|x\| \geq M \). It is known that this number exists if and only if \( F^{-1}(0) \) is compact and, in this case, this is a rational number. The exact computation or the estimation of \( \mathcal{L}_\infty(F) \) from below is a non-trivial problem \([10,17,19,24]\). This number is intimately related with questions about the injectivity of polynomial maps \([5]\) and the equivalence at infinity of polynomial vector fields \([25]\). We give a sufficient condition that implies that there is a uniform \( \text{Łojasiewicz} \) inequality associated to a homotopy of the form \( F + tG, t \in [0, 1] \), where \( G \) denotes another polynomial map, and this gives our result about the invariance of the index (Theorem 8.1). That condition is given in terms of Newton polyhedra and non-degeneracy conditions on maps. We point that inequality (1) can be generalized in many directions, as can be seen in \([11]\), where Newton polyhedra and non-degeneracy are also applied to derive very interesting computations.

In this article we generalize the notion of pre-weighted homogeneous polynomial map \( \mathbb{K}^n \to \mathbb{K}^p \) thus leading to the notion of *strongly adapted map* to a given convenient global Newton polyhedron in \( \mathbb{R}^n \) (Section 4). This is the key idea that allows us to show one of the main results, Theorem 4.4, which gives an estimation of the region in \( \mathbb{R}^n \) determined by the monomials that we call *special* with respect to \( F \) (Definition 4.1). These monomials play a role analogous to the monomials belonging to the integral closure of a given ideal of the ring \( A_n \) of analytic functions \( f : (\mathbb{K}^n, 0) \to \mathbb{K} \). Section 5 is devoted to the proof of this result.

In Sections 6 and 7 we apply Theorem 4.4 to establish a positive lower bound for \( \mathcal{L}_\infty(F) \) (Corollary 6.3 and Proposition 7.3) and to derive a consequence about the injectivity of polynomial maps, which is Corollary 6.5. We remark that in \([2]\) the first author developed a technique to obtain a lower bound for \( \text{Łojasiewicz} \) exponents at infinity that only works in the real case (see Remark 6.6). The proofs in the present paper are mostly self-contained and work simultaneously for real and complex polynomial maps.

Finally in Section 8 we apply the argument of the proof of Theorem 4.4 to obtain a result about the global index of polynomial maps.

## 2. Newton polyhedra at infinity. Preliminary concepts

In this section we expose some basic definitions and results that we will need in subsequent sections.

**Definition 2.1.** Let \( \Gamma_+ \subseteq \mathbb{R}^n_+ \). We say that \( \Gamma_+ \) is a *global Newton polyhedron*, or a *Newton polyhedron at infinity*, if there exists some finite subset \( A \subseteq \mathbb{Z}^n_+ \) such that \( \Gamma_+ \) is equal to the convex hull in \( \mathbb{R}^n \) of \( A \cup \{0\} \).

Let us fix a global Newton polyhedron \( \Gamma_+ \subseteq \mathbb{R}^n_+ \). If \( w \in \mathbb{R}^n \) then we define

\[
\ell(w, \Gamma_+) = \min \{ \langle w, k \rangle : k \in \Gamma_+ \}
\]

\[
\Delta(w, \Gamma_+) = \{ k \in \Gamma_+ : \langle w, k \rangle = \ell(w, \Gamma_+) \}
\]

where we denote by \( \langle \,, \rangle \) the standard scalar product in \( \mathbb{R}^n \). If \( w \in \mathbb{R}^n \setminus \{0\} \), then \( \Delta(w, \Gamma_+) \) is called a *face* of \( \Gamma_+ \). The set \( \Delta(w, \Gamma_+) \) is also called the face of \( \Gamma_+ \) *supported by* \( w \). The hyperplane given by the equation \( \langle w, k \rangle = \ell(w, \Gamma_+) \) is called a *supporting hyperplane* of \( \Gamma_+ \) (this concept can be extended naturally to any convex and closed subset of \( \mathbb{R}^n \)).

The dimension of a face \( \Delta \) of \( \Gamma_+ \), denoted by \( \dim(\Delta) \), is defined as the minimum among the dimensions of the affine subspaces containing \( \Delta \). The faces of \( \Gamma_+ \) of dimension 0 are called the *vertices* of \( \Gamma_+ \) and the faces of \( \Gamma_+ \) of dimension \( n-1 \) are called *facets* of \( \Gamma_+ \). We define the *dimension* of \( \Gamma_+ \) as

\[
\dim(\Gamma_+) = \max \{ \dim(\Delta) : \Delta \text{ is a face of } \Gamma_+ \text{ such that } 0 \notin \Delta \}.
\]