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Growth at infinity and index of polynomial maps ☆

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A B S T R A C T

Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map such that $F^{-1}(0)$ is compact, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then we give a condition implying that there is a uniform bound for the Łojasiewicz exponent at infinity in certain deformations of F. This fact gives a result about the invariance of the global index of F.

1. Introduction

Given a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{-1}(0)$ is finite, in this article we study the problem of determining which monomials can be added to each component function of F leading to a map having the same global index than F. We recall that the *global index of F*, which we denote by $\operatorname{ind}(F)$, is defined as $\operatorname{ind}(F) = \sum_{x \in F^{-1}(0)} \operatorname{ind}_x(F)$, where $\operatorname{ind}_x(F)$ denotes the index, or topological degree, of F at each point $x \in F^{-1}(0)$. The local version of this question, which is analyzed in the articles [1,8,15,20], takes part in the wider problem of determining which monomials in the Taylor expansion of a smooth vector field determine the local phase portrait (see for instance [3], [4] and [28]). The first step in this approach to the study of global indices is the result of Cima, Gasull and Mañosas [7, Proposition 2] on the index of maps whose monomials of maximum degree with respect to some vector of weights have an isolated zero. We call these maps pre-weighted homogeneous (see Definition 7.1 for a precise formulation of this concept).

Apart from [7], our motivation to study global indices comes from the estimation of the Łojasiewicz exponent at infinity of a given polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (see the article of Krasiński [17] for a detailed survey about Łojasiewicz exponents at infinity). This number, which is denoted

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by $\mathcal{L}_\infty(F)$, is defined as the supremum of those real numbers α such that there exist constants $C, M > 0$ such that

$$\|x\|^\alpha \leqslant C\big\|F(x)\big\| \tag{1}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. It is known that this number exists if and only if $F^{-1}(0)$ is compact and, in this case, this is a rational number. The exact computation or the estimation of $\mathcal{L}_\infty(F)$ from below is a non-trivial problem [10,17,19,24]. This number is intimately related with questions about the injectivity of polynomial maps [5] and the equivalence at infinity of polynomial vector fields [25]. We give a sufficient condition that implies that there is a uniform Łojasiewicz inequality associated to a homotopy of the form $F + tG$, $t \in [0, 1]$, where G denotes another polynomial map, and this gives our result about the invariance of the index (Theorem 8.1). That condition is given in terms of Newton polyhedra and non-degeneracy conditions on maps. We point that inequality (1) can be generalized in many directions, as can be seen in [11], where Newton polyhedra and non-degeneracy are also applied to derive very interesting computations.

In this article we generalize the notion of pre-weighted homogeneous polynomial map $\mathbb{K}^n \to \mathbb{K}^p$ thus leading to the notion of *strongly adapted map* to a given convenient global Newton polyhedron in $\mathbb{R}^n$ (Section 4). This is the key idea that allows us to show one of the main results, Theorem 4.4, which gives an estimation of the region in $\mathbb{R}^n$ determined by the monomials that we call *special* with respect to F (Definition 4.1). These monomials play a role analogous to the monomials belonging to the integral closure of a given ideal of the ring $\mathcal{A}_n$ of analytic functions $f : (\mathbb{K}^n, 0) \to \mathbb{K}$. Section 5 is devoted to the proof of this result.

In Sections 6 and 7 we apply Theorem 4.4 to establish a positive lower bound for $\mathcal{L}_\infty(F)$ (Corollary 6.3 and Proposition 7.3) and to derive a consequence about the injectivity of polynomial maps, which is Corollary 6.5. We remark that in [2] the first author developed a technique to obtain a lower bound for Łojasiewicz exponents at infinity that only works in the real case (see Remark 6.6). The proofs in the present paper are mostly self-contained and work simultaneously for real and complex polynomial maps.

Finally in Section 8 we apply the argument of the proof of Theorem 4.4 to obtain a result about the global index of polynomial maps.

2. Newton polyhedra at infinity. Preliminary concepts

In this section we expose some basic definitions and results that we will need in subsequent sections.

Definition 2.1. Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. We say that $\widetilde{\Gamma}_+$ is a *global Newton polyhedron*, or a *Newton polyhedron at infinity*, if there exists some finite subset $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $\widetilde{\Gamma}_+$ is equal to the convex hull in $\mathbb{R}^n$ of $A \cup \{0\}$.

Let us fix a global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. If $w \in \mathbb{R}^n$ then we define

$$\ell(w, \widetilde{\Gamma}_+) = \min\big\{\langle w, k\rangle : k \in \widetilde{\Gamma}_+\big\} \tag{2}$$

$$\Delta(w, \widetilde{\Gamma}_+) = \big\{k \in \widetilde{\Gamma}_+ : \langle w, k\rangle = \ell(w, \widetilde{\Gamma}_+)\big\} \tag{3}$$

where we denote by $\langle\,,\rangle$ the standard scalar product in $\mathbb{R}^n$. If $w \in \mathbb{R}^n \smallsetminus \{0\}$, then $\Delta(w, \widetilde{\Gamma}_+)$ is called a *face* of $\widetilde{\Gamma}_+$. The set $\Delta(w, \widetilde{\Gamma}_+)$ is also called the face of $\widetilde{\Gamma}_+$ *supported by* w. The hyperplane given by the equation $\langle w, k\rangle = \ell(w, \widetilde{\Gamma}_+)$ is called a *supporting hyperplane of* $\widetilde{\Gamma}_+$ (this concept can be extended naturally to any convex and closed subset of $\mathbb{R}^n$).

The *dimension* of a face Δ of Γ_+, denoted by $\dim(\Delta)$, is defined as the minimum among the dimensions of the affine subspaces containing Δ. The faces of $\widetilde{\Gamma}_+$ of dimension 0 are called the *vertices* of $\widetilde{\Gamma}_+$ and the faces of $\widetilde{\Gamma}_+$ of dimension $n-1$ are called *facets* of $\widetilde{\Gamma}_+$. We define the *dimension* of $\widetilde{\Gamma}_+$ as

$$\dim(\widetilde{\Gamma}_+) = \max\big\{\dim(\Delta) : \Delta \text{ is a face of } \widetilde{\Gamma}_+ \text{ such that } 0 \notin \Delta\big\}.$$