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Boundedness of the attraction–repulsion Keller–Segel system

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ABSTRACT

This paper considers the initial–boundary value problem of the attraction–repulsion Keller–Segel model describing aggregation of *Microglia* in the central nervous system in Alzheimer's disease due to the interaction of chemoattractant and chemorepellent. If repulsion dominates over attraction, we show the global existence of classical solution in two dimensions and weak solution in three dimensions with large initial data.

1. Introduction and main results

This paper is concerned with the initial–boundary value problem of the following attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(\chi u\nabla v) + \nabla\cdot(\xi u\nabla w), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad \tau v(x,0)=\tau v_0(x),\quad \tau w(x,0)=\tau w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The model (1.1) was proposed in [18] to describe how the combination of chemicals might interact to produce aggregates of cells. A documented example is the motion of *Microglia* in the central nervous system (CNS) in Alzheimer's disease (AD) which is affected by the interaction of chemoattractant (e.g., β-amyloid) and chemorepellent (e.g., TNF-α) which are secreted by *Microglia*, where the concentrations of *Microglia*, chemoattractant and chemorepellent are

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denoted by $u(x,t)$, $v(x,t)$ and $w(x,t)$ in the model (1.1) respectively. The positive parameters χ and ξ are called the chemosensitivity coefficients, and $\chi, \beta, \gamma, \delta > 0$ are chemical production and depredation rates. It is noted that the chemotaxis model with attractive and repulsive chemicals was also proposed in the paper [22] to interpret the quorum sensing effect in the chemotactic movement.

Proposed first by Keller and Segel [13], the classical (attractive) chemotaxis model was a system of two partial differential equations (i.e. the first two equations of (1.1) with $\xi = 0$) which possesses an apparent Lyapunov functional. This particular structure motivated a vast amount of mathematical studies in the past (see review articles [7,9,27]) and recent studies [3,8,26,28,29], where most of works were focused at whether the solution blows up or not (see some early works in [10,19,20] in this area). On the other hand, for the repulsive Keller–Segel model (i.e. the coupling of first and third equations of (1.1) with $\chi = 0$), a Lyapunov function (which was different from that of the attractive Keller–Segel model) was found in [4] which leads to the global existence of classical solutions in two dimensions and weak solutions in three and four dimensions. Compared to the classical Keller–Segel model, the three-component system of attraction–repulsion Keller–Segel (ARKS) model (1.1) is much harder to analyze due to the lack of an apparent Lyapunov functional. Since after a preliminary result on the linear stability analysis in one-dimensional space in the work [18], no progress has been made until a recent work by Tao and Wang [25] where the main contribution has three folds: (1) $\tau = 0$, the parameter regime of global boundedness and blowup of solutions were successfully identified by the Moser iteration method, which reveals the competing effect of attraction and repulsion plays a central role in determining the dynamics of solutions; (2) when $\tau = 1$ and $\beta = \delta$, numerous clean transformations were introduced to reduce the ARKS model (1.1) to the classical chemotaxis model so that the existing mathematical techniques (like Lyapunov functional) and results could be employed to derive various behaviors of solutions; (3) when $\tau = 1$ and $\beta \neq \delta$, an entropy inequality was provided to establish the time dependent global boundedness of solutions when the initial mass $\int_\Omega u_0 dx$ is small and repulsion prevails (i.e. $\xi\gamma - \chi\alpha > 0$).

The study of [25] leaves two evident gaps in the case of $\tau = 1$ and $\beta \neq \delta$: (a) existence of global solutions with uniform-in-time boundedness or with large data of initial value u_0 if the repulsion dominates; (b) behavior of solutions if the attraction prevails. All the past and current methods (e.g. see [10,19,20, 28,29]) of proving the blowup of solutions of the attractive Keller–Segel model essentially depend on the existence of a Lyapunov functional. It appears to be hopeless at present due to the failure of finding a Lyapunov functional to establish the blowup of solutions for the case where the attraction prevails (i.e. $\xi\gamma - \chi\alpha < 0$). This paper is devoted to explore the questions left in the first gap. We specifically obtain the following main results in the present paper.

Theorem 1.1. *Assume that $0 \le (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ with $u_0 \not\equiv 0$ and $\xi\gamma > \chi\alpha$. Then the ARKS model (1.1) with $\tau = 1$ has a unique nonnegative classical solution $(u, v, w) \in C^0(\bar{\Omega} \times [0,\infty); \mathbb{R}^3) \cap C^{2,1}(\bar{\Omega} \times (0,\infty); \mathbb{R}^3)$ such that $u > 0$ in $\Omega \times (0,\infty)$. Furthermore, there exists a constant C independent of t such that*

$$\left\| u(\cdot, t) \right\|_{L^\infty} \le C.$$

Remark 1.1. In the above theorem, we do not impose the smallness assumption on the initial mass $\int_\Omega u_0(x)dx$ for the global existence of solutions with uniform-in-time bound, which substantially improves the results of [25, Theorem 2.7].

In three dimensions, we introduce the notion of weak solutions to (1.1) to be used later.

Definition 1.1. A global weak solution to (1.1) is a triple of nonnegative functions

$$(u, v, w) \in C\big([0,\infty); weak - L^1\big(\Omega; \mathbb{R}^3\big)\big)$$