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# Quadratures and orthogonality associated with the Cayley transform

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ABSTRACT

In this paper, we are dealing with the approximate calculation of weighted integrals over the whole real line. The method is based in passing to the unit circle by means of the so-called "Cayley transform", $z = \frac{i-x}{i+x}$ and then making use of a Szegő or interpolatory-type quadrature formula on the unit circle, in order to obtain a Gauss-like quadrature rule on the real line. Some properties concerning orthogonality, maximal domains of validity of the quadratures and connections with certain orthogonal rational functions are presented. Finally, some numerical experiments are also carried out.



## 1. Introduction

It is well-known that the Cayley transform $z = \frac{i-x}{i+x}$ maps the right hand half plane onto the upper half plane and the real line $\mathbb{R}$ onto the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and thus, gives rise to a correspondence between certain problems, along with their possible solutions, considered on these contours. In this paper we are dealing with one of such problems: the approximate calculation of weighted integrals either on $\mathbb{R}$ or on $\mathbb{T}$. Let $\sigma$ be a weight function on $(a, b)$, $(-\infty \le a < b \le +\infty)$, i.e. $\sigma > 0$ almost everywhere on $(a, b)$ such that the (polynomial) moments $c_k = \int_a^b x^k \sigma(x)dx$ are finite for every nonnegative integer $k$. Consider the problem of approximation of $I_\sigma(f) = \int_a^b f(x)\sigma(x)dx$ by a quadrature $I_n^\sigma(f) = \sum_{j=1}^n A_j f(x_j)$. A standard way of finding a "good" formula of this type is using the theory of orthogonal polynomials with respect to $\sigma$. Let us denote by $\{Q_k(x)\}_{k=0}^\infty$ the sequence of orthonormal polynomials associated with the weight $\sigma$. If we

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take the zeros of $Q_n$ as the nodes $\{x_j\}_{j=1}^n$ of the quadrature and the corresponding Christoffel numbers $A_j = [\sum_{k=0}^{n-1}(Q_k(x_j))^2]^{-1} > 0$ as the coefficients, we obtain the well known Gaussian formula. For such a formula the relation $I_\sigma(P) = I_n(P)$ holds for every polynomial $P \in \mathbb{P}_{2n-1}$ of degree not exceeding $2n-1$ (here we write $\mathbb{P}_k$ for the space of polynomials of degree less than or equal to $k$ and $\mathbb{P} = \bigcup_{k=0}^{\infty} \mathbb{P}_k$ for the space of all polynomials). Moreover, the positive character of the weights is important for practical, numerical and theoretical reasons.

In fact, we could deal with a more general setting replacing the weight $\sigma$ with a positive measure $\mu$ supported on $(a,b)$, or with $\sigma$ not necessarily positive, or even complex (so that classical orthogonality now becomes meaningless). Again, the zeros of orthogonal polynomials with respect to an appropriate weight function allow us to construct an $n$-point quadrature formula as above which is exact on polynomials in $\mathbb{P}_{n-1}$ and providing excellent approximations to $I_\sigma(f)$, (see e.g. [13]).

In this paper, however, for the sake of clarity of our presentation we will restrict ourselves to the case of an absolutely continuous positive measure $\mu(x)$ i.e. $d\mu(x) = \sigma(x)dx$.

A similar approach involving the approximation of weighted integrals of $2\pi$-periodic functions i.e. integrals of the form $I_\omega(g) = \int_c^d g(\theta)\omega(\theta)d\theta$ where $[c,d]$ is any interval of length $2\pi$ (now and in the sequel we will consider $[-\pi,\pi]$), $g$ and $\omega$ both $2\pi$-periodic and $\omega$ is a weight function on $[-\pi,\pi]$, leads to the so-called Szegő quadrature formulas which are periodic analogs of the Gaussian formulas and are precise for trigonometric polynomials. This means that $I_n^\omega(T) = I_\omega(T)$ for any trigonometric polynomial $T$ of degree at most $n-1$, where

$$I_n^\omega(g) = \sum_{j=1}^n \lambda_j g(\theta_j), \quad \theta_j \neq \theta_k \text{ if } j \neq k \text{ and } \{\theta_j\}_{j=1}^n \subset [-\pi,\pi).$$

We will give more details in Section 2.

When the considered interval $[a,b]$ on $\mathbb{R}$ is finite we can think of it as $[-1,1]$. The connection between the corresponding families of orthogonal polynomials on $[-1,1]$ and $\mathbb{T}$ is well understood (see for example the classical work by Szegő [17, Theorem 11.5]). The connection between quadratures can be found in [2–4]. Zhukowsky's transform[4] $x = \frac{1}{2}(z + \frac{1}{z})$ which maps conformally the interior of the unit circle $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto the exterior of $[-1,1]$ i.e. $\overline{\mathbb{C}}\backslash[-1,1]$ by preserving the point at infinity facilitates these connections.

When trying to extend these connections for an unbounded interval, some partial results were obtained in [8] (see also [9]). For instance, the so-called Rogers–Szegő polynomials ([1], [16, Chapter 1.6]) arise when taking $I = (-\infty,\infty)$ and $\sigma(x) = e^{-\alpha x^2}$, $\alpha > 0$. In this paper, we will further investigate the case $I = (-\infty,\infty)$ discussing quadratures for integrals

$$I_\sigma(f) = \int_{-\infty}^{\infty} f(x)\sigma(x)dx, \tag{1}$$

with $\sigma$ a weight function on $\mathbb{R}$. The particular case $\sigma(x) = \frac{1}{P(x)}$, where $P$ is a positive real polynomial will be considered in Section 5. We use the Cayley transform and reduce (1) to

$$I_\sigma(f) = I_\omega(g) = \int_{-\pi}^{\pi} g(\theta)\omega(\theta)d\theta, \tag{2}$$

with $g$ and $\omega$ both periodic functions so that the Szegő quadrature rules can be used to give an estimation of (2) and consequently of (1). In order to be able to carry out the above approach without paying a high

[4] Also called the Joukowsky transform.