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Approximation in Sobolev spaces by piecewise affine interpolation

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ABSTRACT

Functions in a Sobolev space are approximated directly by piecewise affine interpolation in the norm of the space. The proof is based on estimates for interpolations and does not rely on the density of smooth functions.

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1. Introduction

Functions in classical Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ of weakly differentiable functions, for $p \in [1, \infty[$, can be approximated with respect to the Sobolev norm

$$\|u\|_{W^{1,p}(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |Du|^p + |u|^p \right)^{\frac{1}{p}}$$

by various classes of nicer functions. In order to study Sobolev functions as generalizations of smooth functions, it is natural to approximate them by *smooth Sobolev functions* $C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$ [10] (see also [1, Theorem 3.17], [8], [14, Lemma 2.1.3]) or by *compactly supported smooth function* (see for example [1, Theorem 3.22], [3, Theorem 9.2], [12, Lemma 6.5], [13, Theorem 6.1.10]). In such a way, many properties of Sobolev functions can be proved first by differential calculus arguments for smooth maps and then extended by density to Sobolev functions.

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In the context of numerical resolution of partial differential equations by simplicial finite elements methods, a natural class of nice functions is the space of *piecewise affine functions*. These functions are dense in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ [7, Proposition 2.8] (see also [4, Theorem 3.2.3], [6, Theorem 12.15]).

The usual proof for this statement consists in proving that piecewise affine functions approximate a function in a dense subset of $W^{1,p}(\Omega)$. The latter set can be either a set of smooth functions or a higher-order Sobolev space. Sharp bounds on this approximation of smoother functions, which are known as Bramble–Hilbert lemmas, play an important role in the mathematical study of the convergence of finite element methods [2,4].

This approach in two steps is conceptually disappointing because it solves the problem of approximating by piecewise affine functions by relying on the approximation by smooth functions which is not a priori simpler, and because the diagonal argument hides the construction of the approximation: the approximating functions are piecewise affine functions whose values at vertices of the triangulation are averages in a neighbourhood of the points, and the scale of the averaging and of the triangulation need not be the same.

The goal of this note is to provide a *direct* approximation of Sobolev functions by *interpolation*. Given a function $u : \mathbb{R}^n \to \mathbb{R}$ and a triangulation $\mathcal{S}$ of the Euclidean space $\mathbb{R}^n$ in nondegenerate $(n+1)$-simplices, the *affine interpolant* with respect to $\mathcal{S}$ (also known as *Lagrange interpolant*) of u is the function $\Pi_{\mathcal{S}} u : \mathbb{R}^n \to \mathbb{R}$ such that for every $(n+1)$-simplex $\Sigma \in \mathcal{S}$, its restriction $\Pi_{\mathcal{S}} u|_{\Sigma}$ is affine and at every vertex a of the simplex Σ the values coincide: $(\Pi_{\mathcal{S}} u)(a) = u(a)$.

We shall prove an approximation result that covers the classical Sobolev spaces as well as homogeneous Sobolev spaces.

Theorem 1. *Let $u \in L^q(\mathbb{R}^n)$. If u is weakly differentiable and if $Du \in L^p(\mathbb{R}^n)$, then for every $\varepsilon > 0$ there exists a triangulation $\mathcal{S}$ of $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \bigl|D(u - \Pi_{\mathcal{S}} u)\bigr|^p + \int_{\mathbb{R}^n} |u - \Pi_{\mathcal{S}} u|^q \le \varepsilon.$$

In order to have a well-defined interpolation $\Pi_{\mathcal{S}} u$, we assume in this statement that the function u is *defined at every point* of the space $\mathbb{R}^n$, that is, we do not consider u as an equivalence class in the Lebesgue space $L^p(\mathbb{R}^n)$. It will appear in the proof that the vertices of the triangulation will be Lebesgue points of the function u.

The weakness of such a statement is the dependence of the triangulation $\mathcal{S}$ on the function u. Due to the minimal regularity assumptions on the function u, this dependence is unavoidable. The reader will see in the proof that the triangulation is obtained by dilation and translation from a fixed dilation and that a large set of triangulations can be used to approximate a given function u.

2. Proof of the theorem

In order to prove Theorem 1, we study representation formulas of the affine interpolant. We begin by a Sobolev integral representation formula on a simplex.

Given a convex set $C \subset \mathbb{R}^n$ and point $a \in C$, the associated Minkowski gauge $\gamma_a^C : \mathbb{R}^n \to [0, \infty]$ is defined [5, p. 40], [11, p. 28] at every point $x \in \mathbb{R}^n$ by

$$\gamma_a^C(x) = \inf\bigl\{\lambda \in (0, \infty) : a + \lambda^{-1}(x - a) \in C\bigr\}.$$

We observe that $\gamma_a^C(x) \le 1$ if and only if the point x lies in the closure of the convex set C.

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