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Bohr radius for subordinating families of analytic functions and bounded harmonic mappings

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ABSTRACT

The class consisting of analytic functions f in the unit disk satisfying $f + \alpha z f' + \gamma z^2 f''$ subordinated to some function h is considered. The Bohr radius for this class is obtained when h is respectively convex or starlike. The Bohr radius for analytic functions mapping the unit disk into a concave-wedge domain as well as for bounded harmonic mappings are also established.

1. Introduction

The Bohr inequality describes the size of the sum of the moduli of the terms in the series expansion of a bounded analytic function. Specifically it states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disk $U := \{z : |z| < 1\}$ and $|f(z)| < 1$ for all $z \in U$, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ for all $|z| \leq 1/3$. This inequality was obtained by Bohr [10] in 1914, and the constant $r_0 = 1/3$ is known as the Bohr radius. Bohr actually obtained the inequality for $|z| \leq 1/6$, but subsequently later, Wiener, Riesz and Schur, independently established the sharp inequality for $|z| \leq 1/3$ [16,23,25]. Other proofs have also been given in [17–19].

More generally, the Bohr radius for bounded analytic functions in the unit disk can be paraphrased in terms of its supremum norm, that is, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $\|f\|_\infty = \sup_{|z|<1} |f(z)| < \infty$, then

$$\sum_{n=0}^{\infty} |a_n z^n| \leq \|f\|_\infty$$

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when $|z| \leq 1/3$. Boas and Khavinson [9], and Aizenberg [3,4,6] have extended the inequality to several complex variables. More recently Defant et al. [11] obtained the optimal asymptotic estimate for the n-dimensional Bohr radius on the polydisk U^n.

Operator algebraists have also taken a keen interest in the Bohr inequality, particularly after Dixon [12] used it to settle in the negative a conjecture on Banach algebras. Pursuant to this construction, Paulsen and Singh [18] have extended the Bohr inequality in the context of Banach algebras.

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the Bohr inequality can be put in the form

$$d\left(\sum_{n=0}^{\infty}|a_n z^n|, |a_0|\right) = \sum_{n=1}^{\infty}|a_n z^n| \leq d\big(f(0), \partial U\big), \tag{1}$$

where d is the Euclidean distance. More generally, a class of analytic (or harmonic) functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ mapping U into a domain Ω is said to satisfy a Bohr phenomenon if an inequality of type (1) holds uniformly in $|z| < \rho_0$, $0 < \rho_0 \leq 1$, and for all functions in the class. The notion of the Bohr phenomenon was first introduced in [8] for a Banach space X of analytic functions in the disk U. It was shown that under the usual norm, the Bohr phenomenon does not hold for the Hardy spaces H^p, $1 \leq p < \infty$. However use of a different norm might lead to the occurrence of a Bohr phenomenon. In [8], a characterization of appropriate norms was obtained that yielded the Bohr phenomenon for X.

An important notion in complex function theory is subordination. Given two analytic functions f and g, the function g is subordinate to f, written $g(z) \prec f(z)$, if g is the composition of f with an analytic self-map w of the unit disk with $w(0) = 0$. In the case f is univalent, subordination is equivalent to $g(U) \subset f(U)$ and $g(0) = f(0)$. For additional details on subordination classes, see for example [13, Chapter 6] or [20, p. 35].

To make precise the notion of the Bohr phenomenon for classes of functions, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a given analytic function in U with $f(U) = \Omega$. Denote by $S(f)$ the class of analytic functions g subordinate to f. The class $S(f)$ is said to satisfy a Bohr phenomenon if there is a constant $\rho_0 \in (0, 1]$ satisfying

$$\sum_{n=1}^{\infty}|b_n z^n| \leq d\big(f(0), \partial\Omega\big)$$

for all $|z| < \rho_0$, and for any $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$. The constant ρ_0 is called the Bohr radius.

When f is convex, that is, $f(U)$ is a convex domain, Aizenberg [5, Theorem 2.1] showed that the Bohr radius for $S(f)$ is $\rho_0 = 1/3$, a result which includes (1) when $\Omega = U$. Abu-Muhanna [1, Theorem 1] showed that $S(f)$ has a Bohr phenomenon for f univalent, and that the sharp Bohr radius is $3 - 2\sqrt{2} \cong 0.17157$. Equality is attained for the Koebe function $f(z) = z/(1 - z)^2$. In a recent paper [15], we had studied the Bohr phenomenon for functions mapping the unit disk into the exterior of a compact convex set.

In Section 2, the Bohr radius is obtained for the class of analytic functions mapping U into a concave-wedge domain. This result established a link between the results of Aizenberg [5] and Abu-Muhanna [1]. Section 3 deals with subordinating families to convex or starlike functions. Specifically the class $R(\alpha, \gamma, h)$ consisting of analytic functions f satisfying $f(z) + \alpha z f'(z) + \gamma z^2 f''(z) \prec h(z)$ in U is considered. The Bohr radius is obtained for $R(\alpha, \gamma, h)$ when h is respectively convex or starlike. The final section is devoted to finding the Bohr radius for bounded harmonic mappings in the unit disk. Connections of the results obtained in this paper to several earlier works will also be illustrated.

2. Bohr's radius for concave-wedge domains

A link to the earlier results of Aizenberg [5] and Abu-Muhanna [1] could be established by considering the concave-wedge domains

$$W_\alpha := \left\{w \in \mathbb{C} : |\arg w| < \frac{\alpha\pi}{2}\right\}, \quad 1 \leq \alpha \leq 2. \tag{2}$$