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Higher nonlocal problems with bounded potential

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ABSTRACT

The aim of this paper is to study a class of nonlocal fractional Laplacian equations depending on two real parameters. More precisely, by using an appropriate analytical context on fractional Sobolev spaces due to Servadei and Valdinoci, we establish the existence of three weak solutions for nonlocal fractional problems exploiting an abstract critical point result for smooth functionals. We emphasize that the dependence of the underlying equation from one of the real parameters is not necessarily of affine type.

1. Introduction

This paper is devoted to the following two-parameter nonlocal problem, namely $(P^{\mu,\lambda,h}_{M,K,f})$:

$$\begin{cases} -M\left(\|u\|^2_{X_0}\right)\mathcal{L}_K u = \mu h\left(\displaystyle\int_\Omega \left(\int_0^{u(x)} f(x,t)dt\right)dx - \lambda\right) f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Here and in the sequel, Ω is a bounded domain in $(\mathbb{R}^n, |\cdot|)$ with $n > 2s$ (where $s \in (0,1)$), smooth (Lipschitz) boundary $\partial\Omega$ and Lebesgue measure $|\Omega|$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth, λ and μ are real parameters, M, h are two suitable continuous functions and

$$\|u\|^2_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} \left|u(x) - u(y)\right|^2 K(x-y)dxdy.$$

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Further, $\mathcal{L}_K$ is a nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \big(u(x+y)+u(x-y)-2u(x)\big)K(y)dy \quad \big(x\in\mathbb{R}^n\big)$$

where $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is a function with the properties that:

(k$_1$) $\gamma K\in L^1(\mathbb{R}^n)$, *where* $\gamma(x):=\min\{|x|^2,1\}$;
(k$_2$) *there exists* $\beta>0$ *such that*

$$K(x)\geq\beta|x|^{-(n+2s)},$$

for any $x\in\mathbb{R}^n\setminus\{0\}$;
(k$_3$) $K(x)=K(-x)$, *for any* $x\in\mathbb{R}^n\setminus\{0\}$.

A typical example of the kernel K is given by $K(x):=|x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \quad x\in\mathbb{R}^n.$$

Problem $(P^{\mu,\lambda,h}_{M,K,f})$ is clearly highly nonlocal due to the presence of the fractional operator $\mathcal{L}_K$ and to the map M as well as in the source term f. In our context, to avoid some additional technical difficulties originated by the presence of the term

$$M\bigg(\int_{\mathbb{R}^n\times\mathbb{R}^n} \big|u(x)-u(y)\big|^2 K(x-y)dxdy\bigg),$$

we impose some restrictions on the behavior of M (see Section 3).

This setting includes the Kirchhoff-type problem of the form

$$\begin{cases} -\big(a+b\|u\|^2_{X_0}\big)\mathcal{L}_K u = v(\mu,\lambda,h,f) & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases}$$

where $a,b>0$ and

$$v(\mu,\lambda,h,f) := \mu h\Bigg(\int_\Omega\bigg(\int_0^{u(x)} f(x,t)dt\bigg)dx-\lambda\Bigg)f(x,u),$$

see Remark 3.3.

For completeness, in the vast literature on this subject, we refer the reader to some interesting recent results (in the non-fractional setting) obtained by Autuori and Pucci in [1–3] studying Kirchhoff equations by using different approaches.

We also mention that the same authors studied in [5] the existence and multiplicity of solutions for elliptic equations in $\mathbb{R}^n$, driven by a nonlocal integro-differential operator whose standard prototype is the fractional Laplacian (this work is related to the results on general quasilinear elliptic problems given in [4]).

We seek conditions on the data for which problem $(P^{\mu,\lambda,h}_{M,K,f})$ possesses at least three weak solutions. It is worth pointing out that the variational approach to attack such problems is not often easy to perform;