J. Math. Anal. Appl. 420 (2014) 195–206

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Orbital stability of standing waves of two-component Bose–Einstein condensates with internal atomic Josephson junction

H. Hajaiej

Department of Mathematics, College of Sciences, King Saud University, 11451 Riyadh, Saudi Arabia

ARTICLE INFO

Article history:
Received 6 February 2014
Available online 9 May 2014
Submitted by Goong Chen

Keywords:
Ground state solutions
Standing waves
Orbit
Minimization

ABSTRACT

In this paper, we prove existence, symmetry and uniqueness of standing waves for a coupled Gross–Pitaevskii equations modeling component Bose–Einstein condensates BEC with an internal atomic Josephson junction. We will then address the orbital stability of these standing waves and characterize their orbit.

1. Introduction

The dynamics of a model of a two-component BEC irradiated by an external electromagnetic field are given by the following two coupled nonlinear Schrödinger equations:

$$\left.\begin{aligned} &i\partial_t\psi_j = -\frac{1}{2}\Delta\psi_j + \frac{\gamma^2}{2}|x|^2\psi_j + \beta_{jj}|\psi_j|^2\psi_j + \beta_{ji}|\psi_i|^2\psi_j + \lambda\psi_i + \delta\psi_j \\ &\psi_j(0,x) = \psi_j^0 \end{aligned}\right\} \tag{1.1}$$

$i \neq j = (1,2)$, $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $N = 1,2,3$.

$V(x) = \frac{\gamma^2}{2}|x|^2$ is the trapping potential, $\gamma > 0$.

$\beta_{12} = \beta_{21}$ is the inter-specific scattering length, while β_{11} and β_{22} are the intra ones. λ is the Rabi frequency related to the external electric field. It is the effective frequency to realize the internal atomic Josephson junction by a Raman transition, δ is the detuning constant for the Raman transition. Eq. (1.1) arises in modeling BEC composed of atoms in two hyperfine states in the same harmonic map [1]. Recently, BEC with multiple species have been realized in experiments ([2] and references therein), and many interesting phenomena, which do not appear in the single component BEC, have been observed in the

E-mail address: hhajaiej@ksu.edu.sa.

http://dx.doi.org/10.1016/j.jmaa.2014.04.072
0022-247X/

multi-component BEC. The simplest multi-component BEC can be viewed as a binary mixture, which can be used as a model to produce atomic laser. To our knowledge, the first experiment in this framework has been done quite recently, and has opened the way to many other groups of research who carried out the study of such problems for two-component BEC theoretically and experimentally.

In this paper, we consider a binary BEC model in which there is an irradiation with an electromagnetic field, causing a Josephson-type oscillation between the two species. These condensates are extremely important in physics and nonlinear optics since it is possible to measure the relative phase of one component with respect to the other one [2, Lemma 2.1]. Controlling the relative phase, it is also possible to produce vortices, [7] account is given in [1].

A standing wave for (1.1) is a function $(\psi_1, \psi_2) = (e^{-i\mu_1 t}\Phi_1, e^{-i\mu_2 t}\Phi_2)$ solving this NLS. Thus it satisfies the following $2 \times 2(\mathbb{C})$ elliptic system:

$$\begin{cases} \mu_1\Phi_1 = \left[-\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2 + \delta + \beta_{11}|\Phi_1|^2 + \beta_{12}|\Phi_2|^2\right]\Phi_1 + \lambda\Phi_2 \\ \mu_2\Phi_2 = \left[-\frac{1}{2}\Delta + \frac{\gamma^2}{2}|x|^2 + \delta + \beta_{22}|\Phi_2|^2 + \beta_{12}|\Phi_1|^2\right]\Phi_2 + \lambda\Phi_1 \end{cases} \tag{1.2}$$

Ground state solutions of (1.2) are the minimizes of the following constrained variational problem: For two prescribed real numbers c_1 and c_2

$$\hat{I}_{c_1,c_2} = \inf_{(\Psi_1,\Psi_2)\in\hat{S}_{c_1,c_2}} \hat{E}(\Psi_1,\Psi_2)$$

$$\hat{S}_{c_1,c_2} = \left\{(\Psi_1,\Psi_2) \in \Sigma_{\mathbb{C}}(\mathbb{R}^N) \times \Sigma_{\mathbb{C}}(\mathbb{R}^N) : \int |\Psi_1|^2 = c_1^2 \text{ and } \int |\Psi_2|^2 = c_2^2\right\}.$$

$$\sum(\mathbb{R}^N) = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^2u^2(x)dx < \infty\right\}$$

$$|u|^2_{\sum(\mathbb{R}^N)} = |u|_2^2 + |\nabla u|_2^2 + \big||x|u\big|_2^2$$

$$\sum_{\mathbb{C}}(\mathbb{R}^N) = \left\{z = (u,v) \simeq u + iv : (u,v) \in \sum(\mathbb{R}^N) \times \sum(\mathbb{R}^N)\right\}$$

$$\|z\|^2_{\sum_{\mathbb{C}}(\mathbb{R}^N)} = \|z\|_2^2 + \|\nabla z\|_2^2 + \big\||x|z\big\|_2^2.$$

$$\hat{E}(\Psi) = \hat{E}_0(\Psi_1,\Psi_2) + 2\lambda\int \operatorname{Re}(\Psi_1\overline{\Psi_2})dx, \tag{1.3}$$

with $\bar{f}$ denoting the conjugate part of f and $\operatorname{Re}(f)$ its real one.

$$\hat{E}_0(\psi) = \hat{E}_0(\Psi_1,\Psi_2) = \int_{\mathbb{R}^N} \frac{1}{2}\big[|\nabla\Psi_1|_2^2 + |\nabla\Psi_2|_2^2 + \gamma^2|x|^2\big(|\Psi_1|^2 + |\Psi_2|^2\big)\big] + \delta|\Psi_1|^2 + \frac{1}{2}\beta_{11}|\psi_1|^4 + \frac{1}{2}\beta_{22}|\Psi_2|^4 + \beta_{12}|\Psi_1|^2|\Psi_2|^2dx \tag{1.4}$$

As proved in iii) and iv) of Lemma 2.1 of [2], solving the constrained minimization problem (1.3) is equivalent to the study of the auxiliary minimization problem:

$$\widetilde{\hat{I}}_{c_1,c_2} = \inf_{(\Psi_1,\Psi_2)\in\hat{S}_{c_1,c_2}} \widetilde{\hat{E}}(\Psi_1,\Psi_2) \tag{1.5}$$

where