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Relative weak injectivity of operator system pairs

Angshuman Bhattacharya

Department of Mathematics and Statistics, University of Regina, S4S 0A2, Canada

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ABSTRACT

The concept of a relatively weakly injective pair of operator systems is introduced and studied in this paper, motivated by relative weak injectivity in the C*-algebra category. E. Kirchberg [11] proved that the C*-algebra $C^*(\mathbb{F}_\infty)$ of the free group $\mathbb{F}_\infty$ on countably many generators characterises relative weak injectivity for pairs of C*-algebras by means of the maximal tensor product. One of the main results of this paper shows that $C^*(\mathbb{F}_\infty)$ also characterises relative weak injectivity in the operator system category. A key tool is the theory of operator system tensor products [9,10].

1. Introduction

A pair $(\mathcal{A}, \mathcal{B})$ of unital C*-algebras is a *relatively weakly injective pair* for every unital C*-algebra $\mathcal{C}$, $\mathcal{A} \otimes_{\max} \mathcal{C}$ is a unital C*-subalgebra of $\mathcal{B} \otimes_{\max} \mathcal{C}$. (In particular, one has that $\mathcal{A}$ is a unital C*-subalgebra of $\mathcal{B}$.) It is common to say that $\mathcal{A}$ *is relatively weakly injective in* $\mathcal{B}$ if the pair $(\mathcal{A}, \mathcal{B})$ is a relatively weakly injective pair. Relative weak injectivity for pairs of C*-algebras was introduced by E. Kirchberg [11] and was motivated by the work of E.C. Lance [13] on the weak expectation property for C*-algebras.

The purpose of this paper is to introduce and study a notion of relative weak injectivity for pairs $(\mathcal{S}, \mathcal{T})$ of operator systems $\mathcal{S}$ and $\mathcal{T}$. To do so, one therefore needs to consider operator system tensor products. Although the theory of tensor products [9,10] in the category $\mathcal{O}_1$, whose objects are operator systems and whose morphisms are unital completely positive (ucp) linear maps, shares many similarities with C*-algebraic tensor products, there are some significant differences, particularly when considering the operator system analogue of the maximal C*-algebraic tensor product, $\otimes_{\max}$. With the max tensor product, there are two distinct tensor products (denoted by $\otimes_c$ and $\otimes_{\max}$) in the category $\mathcal{O}_1$ that collapse to the maximal C*-algebraic tensor product on the subcategory of unital C*-algebras and unital $*$-homomorphisms. In this paper an operator system analogue of relative weak injectivity will be developed using the commuting tensor product, $\otimes_c$. Specifically, a pair $(\mathcal{S}, \mathcal{T})$ of operator systems is said to be a *relatively weakly injective pair* if, for every operator system $\mathcal{R}$, $\mathcal{S} \otimes_c \mathcal{R}$ is a unital operator subsystem of $\mathcal{T} \otimes_c \mathcal{R}$.

E-mail address: bhattaca@uregina.ca.

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The C*-algebra $C^*(\mathbb{F}_\infty)$ of the free group $\mathbb{F}_\infty$ on countably infinitely many generators is universal in the sense that every unital separable C*-algebra is a quotient of $C^*(\mathbb{F}_\infty)$. Therefore, it is striking that the C*-algebra $C^*(\mathbb{F}_\infty)$ can be used to characterise both the weak expectation property and relative weak injectivity, as demonstrated by two important theorems of Kirchberg. More precisely, $\mathcal{A}$ has WEP if and only if $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$ [11, Proposition 1.1], and $(\mathcal{A}, \mathcal{B})$ is a relatively weakly injective pair if and only if $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \subset \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty)$ [11, Proposition 3.1].

An operator system analogue of the weak expectation property for C*-algebras – namely the double commutant expectation property – was introduced and studied in [8,10], and it was shown that $C^*(\mathbb{F}_\infty)$ characterises this property. One of the main results of this paper shows that $C^*(\mathbb{F}_\infty)$ also characterises relative weak injectivity of operator system pairs (Theorem 4.1). In addition to establishing some alternate characterisations of relative weak injectivity, the existence of relatively weakly injective pairs $(\mathcal{S}, \mathcal{T})$ in the operator system category will be achieved (in Theorem 4.2) in a manner similar to Kirchberg's result [11, Corollary 3.5] that every unital separable C*-algebra is a unital C*-subalgebra of a unital separable C*-algebra with the weak expectation property. The paper concludes with a selection of examples.

The theory of operator algebraic tensor products is treated in the books [1,17], while operator system tensors products are developed in the papers [9,10]. Standard references for operator systems and completely positive maps are [15,16].

2. The commuting operator system tensor product

If $\mathcal{S}$ and $\mathcal{T}$ are operator systems, then the notation $\mathcal{S} \subset \mathcal{T}$ means that $\mathcal{S}$ is a unital operator subsystem of $\mathcal{T}$. That is, if $1_\mathcal{S}$ and $1_\mathcal{T}$ denote the distinguished Archimedean order units for $\mathcal{S}$ and $\mathcal{T}$ respectively, then $1_\mathcal{S} = 1_\mathcal{T}$. Unless the context is not clear, the order unit for an operator system will be denoted simply by 1.

The algebraic tensor product $\mathcal{S} \otimes \mathcal{T}$ of operator systems $\mathcal{S}$ and $\mathcal{T}$ is a $*$-vector space. An *operator system tensor product structure* on $\mathcal{S} \otimes \mathcal{T}$ is a family $\tau = \{\mathcal{C}_n\}_{n\in\mathbb{N}}$ of cones $\mathcal{C}_n \subset M_n(\mathcal{S} \otimes \mathcal{T})$ such that:

(1) $(\mathcal{S} \otimes \mathcal{T}, \tau, 1_\mathcal{S} \otimes 1_\mathcal{T})$ is an operator system, denoted by $\mathcal{S} \otimes_\tau \mathcal{T}$, in which $1_\mathcal{S} \otimes 1_\mathcal{T}$ is an Archimedean order unit,
(2) $M_n(\mathcal{S})_+ \otimes M_m(\mathcal{T})_+ \subset \mathcal{C}_{nm}$, for all $n, m \in \mathbb{N}$, and
(3) if $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive (ucp) maps, then $\phi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{nm}$ is a ucp map.

Recall that a unital completely positive linear (ucp) map $\phi : \mathcal{S} \to \mathcal{T}$ of operator systems is a *complete order isomorphism* if it is a linear bijection and if both ϕ and ϕ^{-1} are completely positive. If the ucp map ϕ is merely injective, then ϕ is a *complete order injection* if ϕ is a complete order isomorphism of between $\mathcal{S}$ and the operator subsystem $\phi(\mathcal{S})$ of $\mathcal{T}$.

If $\mathcal{S}_1 \subset \mathcal{T}_1$ and $\mathcal{S}_2 \subset \mathcal{T}_2$ are inclusions of operator systems, and if $\iota_j : \mathcal{S}_j \to \mathcal{T}_j$ are the inclusion maps, then for any operator system structures τ and σ on $\mathcal{S}_1 \otimes \mathcal{S}_2$ and $\mathcal{T}_1 \otimes \mathcal{T}_2$, respectively, the notation (as used in [5] also)

$$\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_+ \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$$

expresses the fact that the linear vector-space embedding $\iota_1 \otimes \iota_2 : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2$ is a ucp map $\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \to \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$. That is, $\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_+ \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2$ if and only if $M_n(\mathcal{S}_1 \otimes_\tau \mathcal{S}_2)_+ \subset M_n(\mathcal{T}_1 \otimes_\sigma \mathcal{T}_2)_+$ for every $n \in \mathbb{N}$. If, in addition, $\iota_1 \otimes \iota_2$ is a complete order isomorphism onto its range, then this is denoted by

$$\mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \subset_{\text{coi}} \mathcal{T}_1 \otimes_\sigma \mathcal{T}_2.$$

Thus, $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_\sigma \mathcal{T}$ means $\mathcal{S} \otimes_\tau \mathcal{T} \subset_{\text{coi}} \mathcal{S} \otimes_\sigma \mathcal{T}$ and $\mathcal{S} \otimes_\sigma \mathcal{T} \subset_{\text{coi}} \mathcal{S} \otimes_\tau \mathcal{T}$.