J. Math. Anal. Appl. 420 (2014) 279–313

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Resonances for Dirac operators on the half-line

Alexei Iantchenko [a,*], Evgeny Korotyaev [b]

[a] Malmö högskola, Teknik och samhälle, 205 06 Malmö, Sweden
[b] Mathematical Physics Department, Faculty of Physics, Ulianovskaya 2, St. Petersburg State University, St. Petersburg, 198904, Russia

ARTICLE INFO

Article history:
Received 9 July 2013
Available online 2 June 2014
Submitted by D. O'Regan

Keywords:
Resonances
1D Dirac
Modified Fredholm determinant

ABSTRACT

We consider the 1D Dirac operator on the half-line with compactly supported potentials. We study resonances as the poles of scattering matrix or equivalently as the zeros of modified Fredholm determinant. We obtain the following properties of the resonances: (1) asymptotics of counting function, (2) estimates on the resonances and the forbidden domain.

1. Introduction

Consider the free Dirac operator H_0 acting in the Hilbert space $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$ with the Dirichlet boundary condition at $x = 0$ and given by

$$H_0 f = -i\sigma_2 f' + \sigma_3 m f = \begin{pmatrix} m f_1 & -f_2' \\ f_1' & -m f_2 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \ f_1(0) = 0. \tag{1.1}$$

Here $m > 0$ is the mass and σ_j, $i = 1, 2, 3$, are the Pauli matrices

$$\sigma_0 = I_2, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define the perturbed Dirac operator H by

$$H = H_0 + V = \begin{pmatrix} m + p_1 & -\partial_x + q \\ \partial_x + q & -m + p_2 \end{pmatrix}. \tag{1.2}$$

We consider a perturbation of the form

* Corresponding author.
E-mail addresses: ai@mah.se (A. Iantchenko), korotyaev@gmail.com (E. Korotyaev).

http://dx.doi.org/10.1016/j.jmaa.2014.05.081
0022-247X/

$$V(x) = \begin{pmatrix} p_1 & q \\ q & p_2 \end{pmatrix}(x), \quad x \geqslant 0, \tag{1.3}$$

where p_1, p_2 and q are real-valued functions satisfying

$$p_1, p_2, q \in L^1(\mathbb{R}_+).$$

Later, we shall place further restrictions on these functions.

We recall some well-known spectral properties of the Dirac operators, see for example [27]. The operators H_0, H with Dirichlet condition (1.1) are self-adjoint in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$. The spectrum of H_0 is absolutely continuous and is given by

$$\sigma(H_0) = \sigma_{\mathrm{ac}}(H_0) = \mathbb{R} \setminus (-m, m).$$

The spectrum of H consists of the absolutely continuous part $\sigma_{\mathrm{ac}}(H) = \sigma_{\mathrm{ac}}(H_0)$ and a finite number of simple eigenvalues in the gap $(-m, m)$.

In this paper we will study the scattering resonances. Resonances are the complex numbers associated with the outgoing modes and can be defined as the poles of analytic continuation of the resolvent acting between suitable distribution spaces of distributions (see Definition 2.1 below). From a physicists' point of view, the resonances were first studied by Regge in 1958 (see [30]). Since then, the properties of resonances for the Schrödinger type operators had been the object of intense study and we refer to [34] and [38] for the mathematical approach in the multi-dimensional case and references given there. The resonances were defined by the method of complex scaling under the hypothesis that a real-valued smooth potential extends analytically to a complex conic neighborhood of the real domain at infinity and tends to 0 sufficiently fast there as $x \to \infty$. As result, only local or semi-classical properties of resonances could be derived. In the multi-dimensional Dirac case resonances were studied locally in [1].

We are interested in the global properties of resonances which impose further restrictions on the potential. The potential is supposed to have compact support or, at least, super-exponentially decreasing at infinity. In this context, the resonances for the 1D Schrödinger operator are well studied, see Froese [5], Simon [33], Korotyaev [21], Zworski [37] and references given there. We recall that Zworski [37] obtained the first results about the asymptotic distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. Different properties of resonances were determined in [12] and [24]. Inverse problems (characterization, recovering, plus uniqueness) in terms of resonances were solved by Korotyaev for the Schrödinger operator with a compactly supported potential on the real line [23] and the half-line [21]. The "local resonance" stability problems were considered in [22,28].

Similar questions for Dirac operators are much less studied. However, there are a number of papers dealing with other related problems. We mention few papers on Dirac operators with complex-valued potentials on the line by Syroid [35,36], absorption of eigenvalues by continuous spectrum by Seba [32], Weyl–Titchmarsh function for non-selfadjoint matrix Dirac-type equations by Sakhnovich [31], Dirac operators with point interactions by Carlone, Malamud and Posilicano [2], and a recent paper by Cuenin, Laptev and Tretter [4] on the eigenvalue estimates for non-selfadjoint Dirac operators on the real line. In [25] the estimates of the sum of the negative power of all resonances in terms of the norm of the potential and the diameter of its support are determined.

In [14] we consider the 1D massless Dirac operator on the real line with compactly supported potentials. It is a special kind of the Zakharov–Shabat operator (see [6,29]). There were several reasons to deal with the massless case in a separate paper. Technically, this case is much simpler than the massive Dirac operator studied in the present paper, since in the massless case the Riemann surface consists of two disjoint sheets $\mathbb{C}$. Moreover, the resolvent has a simple representation. The goal of [14] was to give a clear untechnical presentation of ideas which are generalized in the present paper and are further developed in our other