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Solutions to a singularly perturbed supercritical elliptic equation on a Riemannian manifold concentrating at a submanifold ☆

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ABSTRACT

Given a smooth Riemannian manifold $(\mathcal{M}, g)$ we investigate the existence of positive solutions to the equation

$$-\varepsilon^2\Delta_g u + u = u^{p-1} \quad \text{on } \mathcal{M}$$

which concentrate at some submanifold of $\mathcal{M}$ as $\varepsilon \to 0$, for supercritical nonlinearities. We obtain a positive answer for some manifolds, which include warped products. Using one of the projections of the warped product or some harmonic morphism, we reduce this problem to a problem of the form

$$-\varepsilon^2\operatorname{div}_h\big(c(x)\nabla_h u\big) + a(x)u = b(x)u^{p-1},$$

with the same exponent p, on a Riemannian manifold (M, h) of smaller dimension, so that p turns out to be subcritical for this last problem. Then, applying Lyapunov–Schmidt reduction, we establish existence of a solution to the last problem which concentrates at a point as $\varepsilon \to 0$.

1. The main results

Let $(\mathfrak{M}, \mathfrak{g})$ be a compact smooth Riemannian manifold, without boundary, of dimension $m \geq 2$. We consider the problem

$$(\wp_\varepsilon) \quad \begin{cases} -\varepsilon^2\Delta_{\mathfrak{g}} v + v = v^{p-1}, \\ v \in H^1_{\mathfrak{g}}(\mathfrak{M}), \quad v > 0, \end{cases}$$

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where $p > 2$ and ε^2 is a singular perturbation parameter. The space $H^1_{\mathfrak{g}}(\mathfrak{M})$ is the completion of $\mathcal{C}^\infty(\mathfrak{M})$ with respect to the norm defined by $\|v\|^2_{\mathfrak{g}} := \int_{\mathfrak{M}}(|\nabla_{\mathfrak{g}} v|^2 + v^2)d\mu_{\mathfrak{g}}$.

Let $2^*_m := \infty$ if $m = 2$ and $2^*_m := \frac{2m}{m-2}$ if $m \geq 3$ be the critical Sobolev exponent in dimension m. In the subcritical case, where $p < 2^*_m$, solutions to $(\wp_\varepsilon)$ which concentrate at a point are known to exist. In [9] Byeon and Park showed that there are solutions with one peak concentrating at a maximum point of the scalar curvature of $(\mathfrak{M}, \mathfrak{g})$ as $\varepsilon \to 0$. Single-peak solutions concentrating at a stable critical point of the scalar curvature of $(\mathfrak{M}, \mathfrak{g})$ as $\varepsilon \to 0$ were obtained by Micheletti and Pistoia in [20], whereas in [13] Dancer, Micheletti and Pistoia proved the existence solutions with k peaks which concentrate at an isolated minimum of the scalar curvature of $(\mathfrak{M}, \mathfrak{g})$ as $\varepsilon \to 0$. Some results on sign changing solutions, as well as multiplicity results, are also available, see [8,10,15,16,21,26].

We are especially interested in the critical and the supercritical case, where $p \geq 2^*_m$, and in solutions exhibiting concentration at positive dimensional submanifolds of $\mathfrak{M}$ as $\varepsilon \to 0$.

For the analogous equation

$$-\varepsilon^2\Delta v + V(x)v = v^{p-1} \quad \text{in } \Omega, \tag{1.1}$$

in a bounded smooth domain Ω in $\mathbb{R}^n$ with Dirichlet or Neumann boundary conditions, or in the entire space $\mathbb{R}^n$, there is a vast literature concerning solutions concentrating at one or at a finite set of points. Various results concerning concentration at an $(n-1)$-dimensional sphere are nowadays also available, see e.g. [2–5,7,19] and the references therein.

A fruitful approach to produce solutions to Eq. (1.1) which concentrate at other positive dimensional manifolds, for p up to some supercritical exponent, is to reduce it to an equation in a domain of lower dimension. This approach was introduced by Ruf and Srikanth in [24], where the reduction is given by a Hopf map. Reductions may also be performed by means of other maps which preserve the laplacian, or by considering rotational symmetries, or by a combination of both, as has been recently done in [1,11,12,17,18,22,23,27] for different problems.

Next, we describe some of these reductions in the context of Riemannian manifolds. For simplicity, from now on we consider all manifolds to be smooth, compact and without boundary.

1.1. Harmonic morphisms

Let $(\mathfrak{M}, \mathfrak{g})$ and (M, g) be Riemannian manifolds of dimensions m and n respectively, and let $\pi : \mathfrak{M} \to M$ be a horizontally conformal submersion with dilation $\lambda : \mathfrak{M} \to (0, \infty)$, i.e. its differential $d\pi_x$ satisfies $g(d\pi_x X, d\pi_x Y) = \lambda^2(x)\mathfrak{g}(X, Y)$ for any two tangent vectors X, Y to $\mathfrak{M}$ at x which are orthogonal to the fiber through x. π is called a *harmonic morphism* if it satisfies the equation

$$(n-2)\mathcal{H}(\nabla_{\mathfrak{g}} \ln \lambda) + (m-n)\kappa^{\mathcal{V}} = 0, \tag{1.2}$$

where $\kappa^{\mathcal{V}}$ is the mean curvature of the fibers and $\mathcal{H}$ is the projection of the tangent space of $\mathfrak{M}$ onto the space orthogonal to the fibers of π, see [6]. Typical examples are the Hopf fibrations

$$\mathbb{S}^m \to \mathbb{R}P^m, \qquad \mathbb{S}^{2m+1} \to \mathbb{C}P^m, \qquad \mathbb{S}^{4m+3} \to \mathbb{H}P^m, \qquad \mathbb{S}^{15} \to \mathbb{S}^8,$$

which are Riemannian submersions (i.e. $\lambda \equiv 1$) with totally geodesic fibers $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ respectively, see [6, Examples 2.4.14–2.4.17]. So they trivially satisfy (1.2).

The main property of a harmonic morphism is that it preserves the Laplace–Beltrami operator, i.e. it satisfies $\Delta_{\mathfrak{g}}(u \circ \pi) = \lambda^2[(\Delta_g u) \circ \pi]$ for every $\mathcal{C}^2$-function $u : M \to \mathbb{R}$. The following proposition is an immediate consequence of this fact.