J. Math. Anal. Appl. 420 (2014) 348–363

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The globally hyperbolic metric splitting for non-smooth wave-type space-times

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ARTICLE INFO

Article history:
Received 4 October 2013
Available online 4 June 2014
Submitted by R. Popovych

Keywords:
Generalized functions
Regularization
Lorentzian manifolds
Non-smooth space-times
Wave-type space-times

ABSTRACT

We investigate a generalization of the so-called metric splitting of globally hyperbolic space-times to non-smooth Lorentzian manifolds and show the existence of this metric splitting for a class of wave-type space-times. The approach used is based on smooth approximations of non-smooth space-times by families (or sequences) of globally hyperbolic space-times. In the same setting we indicate as an application the extension of a previous result on the Cauchy problem for the wave equation.

1. Introduction

We investigate causal properties, especially global hyperbolicity, of wave-type space-times. The relevance of global hyperbolicity in general relativity is due to its role as the strongest established causality condition, in particular in the context of Cauchy problems and singularity theorems. Several equivalent conditions of global hyperbolicity have been investigated and developed, one of the first was existence of a Cauchy hypersurface and the most recent breakthrough was the proof the so-called *metric splitting* (cf. [3]; see also the discussion and Theorem 2.4 in Section 2). To initiate research for an extension of global hyperbolicity to the situation of non-smooth space-times this article aims at providing a case study, thereby also describing the explicit form of the metric splitting in the smooth case. For an overview of wider applications in general relativity of non-smooth Lorentzian metrics with techniques similar to the methods used here we refer to [25].

By wave-type space-times we mean a generalization of plane waves, the so-called N-fronted waves with parallel rays (NPWs) or general plane fronted waves (PFWs). These space-times are given as a product $M = N \times \mathbb{R}^2$, with metric

$$l = \pi^*(h) + 2dudv - a(.,u)du^2, \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jmaa.2014.05.086

where h denotes the metric of an arbitrary connected Riemannian manifold (N, h), $\pi: M \to N$ is the projection ($\pi^*(h)$ denotes the pullback under the projection of h to M) and u, v are global null-coordinates on the two-dimensional Minkowski space $\mathbb{R}^2_1$. Moreover $a: N \times \mathbb{R} \to \mathbb{R}$ is the so-called profile function, which we allow to be non-smooth. Locally in coordinates $x^1, \dots, x^n$ on N at $(x, u, v) \in M$ the metric l can be written as

$$l_{(x,u,v)} = \sum_{i,j=1}^{n} h_{ij} dx^i dx^j + 2dudv - a(x, u)du^2,$$

where h_{ij} denote the metric coefficients of h with respect to $x^1, \dots, x^n$.

NPWs were introduced by Brinkmann in the context of conformal mappings of Einstein spaces [4]. Recently their geometric properties and causal structure were studied in [6,12,7,13] (under the notion of general plane fronted waves – PFW). Due to the geometric interpretation of N as the wave surface of these waves (cf. [23]), it seems more natural to call them N-fronted waves, rather than plane-fronted waves. Note that plane-fronted waves with parallel rays (pp-waves) (cf. [14, Ch. 17]) are a special case of NPWs. In this case $N = \mathbb{R}^2$ with the Euclidean metric.

It turns out (in the classical setting where the metric is smooth) that the behavior of a at spatial infinity, i.e., for "large x" is decisive for many of the global properties of NPWs. In order to formulate precise statements denote by d^h the Riemannian distance function on (N, h) and recall that a is said to behave *subquadratically at spatial infinity*, if there exist a point $\bar{x} \in N$, continuous non-negative functions R_1, $R_2: \mathbb{R} \to (0, \infty)$ and a continuous function $p: \mathbb{R} \to (0, 2)$ such that for all $(x, u) \in N \times \mathbb{R}$

$$a(x, u) \le R_1(u) d^h(x, \bar{x})^{p(u)} + R_2(u). \tag{1.2}$$

Similarly we say that a behaves at most quadratically if $p \le 2$. In [12] it has been shown that the causality of NPWs depends crucially on the exponent p in (1.2), with $p = 2$ being the critical case, which includes classical plane waves that are known to be strongly causal but not globally hyperbolic (cf. [22]). In particular, NPWs are causal but not necessarily distinguishing, they are strongly causal if a behaves at most quadratically at spatial infinity and they are globally hyperbolic if a is subquadratic and N is complete. Similarly the global behavior of geodesics in NPWs is governed by the behavior of a at spatial infinity. From the explicit form of the geodesic equations it follows (see [6, Thm. 3.2]) that a NPW is geodesically complete if and only if N is complete and

$$D^N_{\dot{\xi}} \dot{\xi} = \frac{1}{2} \nabla_x a(\xi, \alpha) \tag{1.3}$$

has complete trajectories for all $\alpha \in \mathbb{R}$, i.e., the solutions of (1.3) can be defined on all of $\mathbb{R}$. Here $D^N_{\dot{\xi}}$ is the induced covariant derivative on N and ∇_x denotes the metric gradient with respect to h. Applying classical results on complete vector fields (e.g. [1, Thm. 3.7.15]) completeness of M follows for autonomous a (i.e., independent of u) in case $-a$ grows at most quadratically at spatial infinity.

When discussing the case of non-smooth profile function a we will also employ the nonlinear theory of generalized functions in the sense of Colombeau, standard references are [9,10,20,15]. Our framework is the so-called special Colombeau algebra $\mathcal{G}$ (denoted by $\mathcal{G}^s$ in [15]) and we briefly recall the basic constructions. Let M be a smooth manifold. *Colombeau generalized functions* on M are defined as equivalence classes $u = [(u_\varepsilon)_\varepsilon]$ of nets of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(M)$ (*regularizations*) subjected to asymptotic norm conditions with respect to $\varepsilon \in (0, 1]$ for their derivatives on compact sets. More precisely, we have

- moderate nets $\mathcal{E}_{\mathrm{M}}(M)$: $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)^{(0,1]}$ such that for any compact subset $K \subseteq M$, $l \in \mathbb{N}_0$, and vector fields $X_1, \dots, X_l$ on M there exists $p \in \mathbb{R}$ such that

$$\|X_l \cdots X_1 u_\varepsilon\|_{L^\infty(K)} = O(\varepsilon^{-p}) \quad (\varepsilon \to 0);$$