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Inequalities for modified Bessel functions and their integrals

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ABSTRACT

Simple inequalities for some integrals involving the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ are established. We also obtain a monotonicity result for $K_\nu(x)$ and a new lower bound, that involves gamma functions, for $K_0(x)$.

1. Introduction and preliminary results

In the developing Stein's method for Variance-Gamma distributions, Gaunt [5] required simple bounds, in terms of modified Bessel functions, for the integrals

$$\int_0^x \mathrm{e}^{\beta t} t^\nu I_\nu(t)\,\mathrm{d}t \quad \text{and} \quad \int_x^\infty \mathrm{e}^{\beta t} t^\nu K_\nu(t)\,\mathrm{d}t,$$

where $x > 0$, $\nu > -1/2$ and $-1 < \beta < 1$. Closed form expressions for these integrals, in terms of modified Bessel functions and the modified Struve function $\mathbf{L}_\nu(x)$, do in fact exist for the case $\beta = 0$. For $z \in \mathbb{C}$ and $\nu \in \mathbb{C}$, let $\mathscr{L}_\nu(z)$ denote $I_\nu(z)$, $\mathrm{e}^{\nu\pi i}K_\nu(z)$ or any linear combination of these functions, in which the coefficients are independent of ν and z. From formula 10.43.2 of Olver et al. [13] we have, for $\nu \neq -1/2$,

$$\int z^\nu \mathscr{L}_\nu(z)\,\mathrm{d}z = \sqrt{\pi}2^{\nu-1}\Gamma(\nu+1/2)z\big(\mathscr{L}_\nu(z)\mathbf{L}_{\nu-1}(z) - \mathscr{L}_{\nu-1}(z)\mathbf{L}_\nu(z)\big). \tag{1.1}$$

Whilst formula (1.1) holds for complex-valued z and ν, throughout this paper we shall restrict our attention to the case of real-valued z and ν. There are no closed form expressions in terms of modified Bessel and Struve functions in the literature for the integrals for the case $\beta \neq 0$. Moreover, even in the case $\beta = 0$ the expression on the right-hand side of formula (1.1) is a complicated expression involving the modified Struve

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function $\mathbf{L}_\nu(x)$. This provides the motivation for establishing simple bounds, in terms of modified Bessel functions, for the integrals defined in the first display.

In this paper we establish, through the use of elementary properties of modified Bessel functions and straightforward calculations, simple bounds, that involve modified Bessel functions, for the integrals given in the first display. Our bounds prove to be very useful when applied to calculations that arise in the study of Stein's method for Variance-Gamma distributions. We also obtain a monotonicity result and bound for the modified Bessel function of the second kind $K_\nu(x)$, as well as a simple but remarkably tight lower bound for $K_0(x)$. These bounds are, again, motivated by the need for such bounds in the study of Stein's method for Variance-Gamma distributions. However, the bounds obtained in this paper may also prove to be useful in other problems related to modified Bessel functions; see, for example, Baricz and Sun [4] in which inequalities for modified Bessel functions of the first kind were used to obtain lower and upper bounds for integrals of involving modified Bessel functions of the first kind. Throughout this paper we make use of some elementary properties of modified Bessel functions and these are stated in Appendix A.

2. Inequalities for integrals involving modified Bessel functions

Before presenting our first result concerning inequalities for integrals of modified Bessel functions, we introduce some notation for the repeated integral of the function $\mathrm{e}^{\beta x}x^\nu I_\nu(x)$, which will be used in the following theorem. We define

$$I_{(\nu,\beta,0)}(x)=\mathrm{e}^{\beta x}x^\nu I_\nu(x), \qquad I_{(\nu,\beta,n+1)}(x)=\int_0^x I_{(\nu,\beta,n)}(y)\,\mathrm{d}y, \quad n=0,1,2,3,\ldots. \tag{2.1}$$

With this notation we have:

Theorem 2.1. *Let $0\le\gamma<1$, then the following inequalities hold for all $x>0$*

$$\int_0^x t^\nu I_\nu(t)\,\mathrm{d}t > x^\nu I_{\nu+1}(x), \quad \nu>-1, \tag{2.2}$$

$$\int_0^x t^\nu I_\nu(t)\,\mathrm{d}t < x^\nu I_\nu(x), \quad \nu\ge 1/2, \tag{2.3}$$

$$I_{(\nu,0,n+1)}(x) < I_{(\nu,0,n)}(x), \quad \nu\ge 1/2, \tag{2.4}$$

$$I_{(\nu,-\gamma,n)}(x) \le \frac{1}{(1-\gamma)^n}\mathrm{e}^{-\gamma x} I_{(\nu,0,n)}(x), \quad \nu\ge 1/2,\ n=0,1,2,\ldots, \tag{2.5}$$

$$\int_0^x t^\nu I_{\nu+n}(t)\,\mathrm{d}t < \frac{2(\nu+n+1)}{2\nu+n+1}x^\nu I_{\nu+n+1}(x), \quad \nu>-1/2,\ n\ge 0, \tag{2.6}$$

$$I_{(\nu,0,n)}(x) < \left\{\prod_{k=1}^n \frac{2\nu+2k}{2\nu+k}\right\} x^\nu I_{\nu+n}(x), \quad \nu\ge 0,\ n=1,2,3,\ldots, \tag{2.7}$$

$$I_{(\nu,-\gamma,n)}(x) < \frac{1}{(1-\gamma)^n}\left\{\prod_{k=1}^n \frac{2\nu+2k}{2\nu+k}\right\}\mathrm{e}^{-\gamma x} x^\nu I_{\nu+n}(x), \quad \nu\ge 1/2,\ n=1,2,3,\ldots.$$