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Positive solutions to a fourth-order elliptic problem by the Lusternik–Schnirelmann category ☆

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ABSTRACT

In this paper we consider the fourth-order problem

$$\begin{cases} \Delta^2 u = \mu |u|^{s-1}u + |u|^{2_*-2}u & \text{in } \Omega, \\ u, -\Delta u > 0 \quad \text{in } \Omega, & u, \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 5$ and $2_* = 2N/(N-4)$. We assume $2 \leq s+1 < 2_*$ in case $N \geq 8$ and $2_* - 2 < s+1 < 2_*$ for the critical dimensions $N = 5, 6, 7$. Then we prove that if Ω has a rich topology, described by its Lusternik–Schnirelmann category, then the problem has multiple solutions, at least as many as $\operatorname{cat}_\Omega(\Omega)$, in case the parameter $\mu > 0$ is sufficiently small.

1. Introduction

In this paper we consider the fourth-order elliptic problem under Navier boundary conditions

$$\begin{cases} \Delta^2 u = \mu |u|^{s-1}u + |u|^{2_*-2}u & \text{in } \Omega, \\ u, -\Delta u > 0 \quad \text{in } \Omega, & u, \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 5$, stands for a bounded smooth domain, $\mu > 0$, $2 \leq s+1 < 2_*$ and $2_* = 2N/(N-4)$ is the critical Sobolev exponent for the embedding $E(\Omega) := H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{2_*}(\Omega)$.

Throughout this paper, $\mu_1(\Omega)$ stands for the first eigenvalue of the problem

$$\begin{cases} \Delta^2 u = \mu u & \text{in } \Omega, \\ u, \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

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The first result in this paper is about the existence of a solution to (P) and our main result shows that if Ω has a rich topology, described by its Lusternik–Schnirelmann category, then multiple solutions to problem (P) exist provided $\mu > 0$ is sufficiently small.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 5$, be a bounded smooth domain and $\mu > 0$. In case $s = 1$ we assume $0 < \mu < \mu_1(\Omega)$. If $N \geq 8$ and $2 \leq s + 1 < 2_*$, or $N = 5, 6, 7$ and $2_* - 2 < s + 1 < 2_*$, then (P) has a classical solution.*

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $N \geq 5$. Assume $2 \leq s + 1 < 2_*$ if $N \geq 8$ and $2_* - 2 < s + 1 < 2_*$ for the critical dimensions $N = 5, 6, 7$. Then there exists $\bar{\mu} > 0$ such that for every $0 < \mu < \bar{\mu}$ the problem (P) has at least $\operatorname{cat}_\Omega(\Omega)$ classical solutions.*

We first observe that, cf. [27], if $2 \leq s+1 < 2_*$ and $\mu \leq 0$ then (P) has no solution in starshaped domains $\Omega \subset \mathbb{R}^N$, $N \geq 5$. If $s = 1$, then the condition $\mu < \mu_1(\Omega)$ is also necessary for the existence of a solution for a general bounded domain Ω. We recall that in [27], assuming that Ω is a general bounded regular domain in $\mathbb{R}^N$, $N \geq 8$, $s = 1$ and $\mu \in (0, \mu_1(\Omega))$, it is proved that (P) has a solution. Gazzola et al. [13] proved that $N = 5, 6, 7$ are the critical dimensions in the sense that (P) has no solution if $\mu > 0$ is small enough, $s = 1$ and Ω is an open ball in $\mathbb{R}^N$; see also [23,17].

We mention that Theorem 1.2 has been proved by Alves and Figueiredo [2] for the context of the corresponding problem under Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u = \mu |u|^{s-1} u + |u|^{2_*-2} u & \text{in } \Omega, \\ u, \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{P1}$$

Theorem 1.1 in [2]. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $N \geq 5$. Assume $2 \leq s+1 < 2_*$ if $N \geq 8$ and $2_* - 2 < s + 1 < 2_*$ for the critical dimensions $N = 5, 6, 7$. Then there exists $\bar{\mu} > 0$ such that for every $0 < \mu < \bar{\mu}$ the problem (P1) has, at least, $\operatorname{cat}_\Omega(\Omega)$ nontrivial classical solutions.*[1]

There is a suitable difference between [2, Theorem 1.1] and Theorem 1.2 in this paper, which involves a qualitative property of the solutions. We benefit from a strong comparison principle which holds for the biharmonic operator under Navier boundary condition. Indeed we know that (P) is equivalent to

$$\Delta^2 u = \mu \big(u^+\big)^s + \big(u^+\big)^{2_*-1} \quad \text{in } \Omega, \qquad u, \Delta u = 0 \quad \text{on } \partial\Omega, \qquad u \not\equiv 0 \quad \text{in } \Omega.$$

Meanwhile, no information about the sign of the solutions of (P1) is provided in [2, Theorem 1.1] since, as is well known, such a comparison principle for the biharmonic operator under Dirichlet boundary conditions does not exist for general domains; cf. [14, Chapter 5].

The general structure of our proof for Theorem 1.2 resembles that for the proof of [2, Theorem 1.1], which indeed has been applied for the corresponding problem with second order elliptic equations and involves arguments based on the Lusternik–Schnirelmann category; cf. Lazzo [18] and Rey [25]. However, to carry out such procedure for the context of problem (P), we face some distinct difficulties and some different arguments are needed. For instance, we have the inclusion $H_0^2(\Omega) \subset \mathcal{D}^{2,2}(\mathbb{R}^N)$ as well as $H_0^1(\Omega) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$, by simple extension by zero outside Ω. But, in general, the extension by zero outside Ω of a function in $E(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$ does not lie in $\mathcal{D}^{2,2}(\mathbb{R}^N)$. Then we employ in parts of our proof a kind of *extension*

[1] There is a misprint in [2, Theorem 1.1] and, according to the notation in that paper, the condition $2_{**} - 2 \leq q < 2_{**}$ might be read as $2_{**} - 2 < q < 2_{**}$. The powers $s + 1$ and 2_* in this paper correspond, respectively, to the powers q and 2^{**} in [2]. See the proof of Lemma 2.4 ahead for more details.