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On essential entropy-carrying sets

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A R T I C L E I N F O

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A B S T R A C T

In this paper we study properties of essential entropy-carrying sets of a continuous map on a compact metric space. If $f : X \to X$ is continuous on a compact metric space X, then the intersection of all essential entropy-carrying sets of f may or may not be an essential entropy-carrying set of f. When this intersection is an essential entropy-carrying set we denote it by $E(f)$, the least essential entropy-carrying set, otherwise we say that $E(f)$ does not exist. We present an example where $E(f)$ does not exist but also find a sufficient condition for $E(f)$ to exist. If f is a piecewise monotone map, we show that $E(f)$ exists and is the finite union of the entropy-carrying sets in the Nitecki Decomposition of the nonwandering set of f intersected with the closure of the periodic points of f. When $E(f)$ exists we study how it relates to other entropy-carrying sets of f including subsets of itself.

1. Introduction

Suppose $f : X \to X$ is a continuous map on a compact metric space X. The topological entropy of f, denoted $h(f)$ and originally defined by Adler, Konheim, and McAndrew [1], measures the dynamical complexity of f. A closed f-invariant set $S \subseteq X$ is called an *entropy-carrying* set of f if $h(f) = h(f|_S)$. For instance, Bowen [4] proved that the nonwandering set $\Omega(f)$ of f is an entropy-carrying set of f. In [11] the author introduced the idea of an essential entropy-carrying set in the following way. An entropy-carrying set $S \subseteq X$ is an *essential* entropy-carrying set of f if for any other entropy-carrying set Y of f, f has an entropy-carrying set $Z \subseteq S \cap Y$. Let $\mathcal{E}(f)$ denote the collection of all entropy-carrying sets of f and let $\mathcal{E}_{ess}(f)$ denote the collection of all essential entropy-carrying sets of f.

An essential entropy-carrying set of f is "indispensable" for producing the complexity of f since it meets every other entropy-carrying set of f in an entropy-carrying set of f. The motivation for considering essential entropy-carrying sets in [11] was to study how topological entropy changes when a continuous map $f : [0, 1] \to [0, 1]$ is modified or perturbed. For example, the following lemma was proven.

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Lemma 1. *(See [11].) Suppose $f : [0,1] \to [0,1]$ is continuous, $h(f) > 0$, and $X \subseteq [a,b] \subseteq [0,1]$ is an essential entropy-carrying set of f. If $F : [0,1] \to [0,1]$ is continuous, $F = f$ on $[0,a] \cup [b,1]$, $F([a,b]) \subseteq [a,b]$, and $h(F|_{[a,b]}) = 0$, then $h(F) < h(f)$.*

One of the main themes of this paper is to consider the existence of an essential entropy-carrying set $E(f)$ of f which is contained in every essential entropy-carrying set of f. When such a set $E(f)$ exists we call it the *least* essential entropy-carrying set of f. An example will show that some maps do not have a least essential entropy-carrying set. However it will be shown that a piecewise monotone interval map always has a least essential entropy-carrying set $E(f)$.

A continuous map $f : I \to I$ where $I \subseteq \mathbb{R}$ is a compact interval is *piecewise monotone* if I has a finite partition into subintervals such that f is monotone on each subinterval. Let $P(f)$ denote the set of periodic points of f. For piecewise monotone maps, Nitecki [10] found a decomposition of $\Omega(f)$ into closed invariant sets $\widetilde{\Omega}_1, \ldots, \widetilde{\Omega}_N$ where $N \leq \infty$ such that $\Omega_i = \widetilde{\Omega}_i \cap \mathrm{cl}(P(f))$ is an entropy-carrying set for only a finite number of i's. It will be shown for piecewise monotone maps that $E(f)$ is the union of these entropy-carrying Ω_i's. The proof also uses the results of Hofbauer [7–9] that f has only a finite number of ergodic maximal measures.

For a piecewise monotone map f we will prove that a nested intersection of essential entropy-carrying sets of f is an essential entropy-carrying set of f. Under this type of condition as a premise, it will be shown for a continuous map $f : X \to X$ of a compact metric space X that $E(f)$ exists. Using the piecewise monotone case as motivation, $E(f)$ could have proper entropy-carrying subsets which are not essential entropy-carrying sets. Therefore we find a sufficient condition for a closed invariant subset of $E(f)$ to be an entropy-carrying set of f.

The following is an outline of this paper. Section 2 reviews the topological entropy of a continuous map on a compact metric space and some properties of entropy-carrying sets. Section 3 studies essential entropy-carrying sets and considers the existence of a least essential entropy-carrying set $E(f)$ for a map f, especially for piecewise monotone maps. When $E(f)$ exists, Section 4 studies how it relates to other entropy-carrying sets of f, including proper subsets of itself.

2. Topological entropy and entropy-carrying sets

Let $f : X \to X$ be continuous where X is a compact metric space with metric d. The *topological entropy* of f, denoted $h(f)$, was originally defined by Adler, Konheim, and McAndrew [1]. An alternate construction was given by Bowen [5] using (n,ϵ)-*separating sets* to define $h(f)$. For $k \geq 1$, let $f^k : X \to X$ be the map where $f^k = f \circ f \circ \ldots \circ f$ (k f's in the composition). $f^0 : X \to X$ will be the identity map. Suppose $\epsilon > 0$ and $n \in \mathbb{Z}^+$ are fixed. $S \subseteq X$ is an (n,ϵ)-*separating set* for f if $x, y \in S$ and $x \neq y$ imply $d(f^k(x), f^k(y)) > \epsilon$ for some $k \in \{0, 1, \ldots, n-1\}$. $s(n,\epsilon,f)$ will denote the maximum cardinality of (n,ϵ)-separating sets for f. Define

$$h(\epsilon, f) = \limsup_{n\to\infty} \frac{\log s(n,\epsilon,f)}{n}.$$

Then by Bowen [5], $h(f) = \lim_{\epsilon\to 0^+} h(\epsilon, f)$.

Next we review two classical results about topological entropy that will be used in this paper. We recall that $S \subseteq X$ is an *invariant set* of f if $f(S) \subseteq S$.

Theorem 2. *(See [1].) If A_1 and A_2 are closed invariant subsets of X such that $X = A_1 \cup A_2$, then $h(f) = \max\{h(f|_{A_1}), h(f|_{A_2})\}$.*

Theorem 3. *(See [1].) If $A \subseteq X$ is a closed invariant set then $h(f|_A) \leq h(f)$.*