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Homogenization for rigid suspensions with random velocity-dependent interfacial forces

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ABSTRACT

We study suspensions of solid particles in a viscous incompressible fluid in the presence of random velocity-dependent interfacial forces. The flow at a small Reynolds number is modeled by the Stokes equations, coupled with the motion of rigid particles arranged in a periodic array. The objective is to perform homogenization for the given suspension and obtain an equivalent description of a homogeneous (effective) medium, the macroscopic effect of the interfacial forces and the effective viscosity are determined using the analysis on a periodicity cell. In particular, the solutions $\boldsymbol{u}_\omega^\epsilon$ to a family of problems corresponding to the size of microstructure ϵ and describing suspensions of rigid particles with random surface forces imposed on the interface, converge H^1-weakly as $\epsilon \to 0$ a.s. to a solution of a Stokes homogenized problem, with velocity dependent body forces. A corrector to a homogenized solution that yields a strong H^1-convergence is also determined. The main technical construction is built upon the Γ-convergence theory.

1. Introduction

Flows of incompressible fluids that carry rigid particles also known as *particulate flows* are essential parts of many engineering and environmental processes (i.e. particle sedimentation, fluidization and conveying) and are commonly encountered in many applications and fundamental fluid mechanics. The complexity of mechanisms that govern fluid–particle and particle–particle interactions makes the numerical simulation of these flows be one of the most challenging problems in computational fluid dynamics. Many analytical, numerical, and experimental studies have been performed during the past decades, however still much more research is needed for the fundamental understanding of these complex heterogeneous media.

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For instance, the *rheological behavior* of particulate flows has been extensively studied for over a century. Up to date the most investigated concentration regime of particles is the dilute one in which the hydrodynamic interactions between inclusions are normally ignored as the interparticle distance exceeds the range of the flows that appear due to the particle motion. Hence, it is possible in this case to isolate the effect of particle–fluid interactions on the effective behavior of suspensions, as was done in the pioneering work by Einstein [15] where the asymptotic study of the effective viscosity μ^* of a suspension of rigid particles in a small concentration regime was carried out. A simple approximation to the viscosity μ^* in volume fraction ϕ of rigid, neutrally buoyant, spheres in the suspension had been formally derived there (see also [16]):

$$\mu^* \simeq \mu\left(1 + \frac{5}{2}\phi\right) \quad \text{as } \phi \to 0, \tag{1.1}$$

where μ is the viscosity of the ambient fluid. The subsequent extension to the ellipsoidal particles was done in [22]. Despite the seemingly simple linear relation (1.1) between μ^* and ϕ, the rigorous justification of the Einstein's formula (1.1) has been carried out over a century later, in [20] by constructing explicit upper and lower bounds on the effective viscosity μ^*.

A pairwise particle interaction in the dilute regime has been first taken into account in [5] to formally derive an $O(\phi^2)$-correction to the Einstein's formula. Such an $O(\phi^2)$-correction in the case of a periodic suspension of spherical fluid drops of viscosity $\eta \to \infty$ in a fluid of viscosity μ was rigorously proven, also very recently, in [1] using techniques different from [20] based on the layer-potential approach previously developed by the authors. For a more detailed description of asymptotic studies for the effective viscosity of dilute suspensions we refer a reader to [1] and references therein.

Another "extreme" regime of particle concentration is the dense packing of particles in a suspension when the typical interparticle distances are much smaller than their sizes. In such a regime the effective rheological properties of suspensions exhibit singular behavior as the characteristic distance between particles tends to zero (or equivalently, as ϕ tends to the maximal packing volume). Such densely-packed particulate flows were also extensively studied both numerically and analytically (see e.g. [17,19,30,33] and references therein) over the past decades. For example, in [17] a local formal asymptotic analysis based on a pair of closely spaced particles showed that the effective viscosity, described by the viscous dissipation rate, exhibits a blow-up of order $O(\delta^{-1})$, where δ is the distance between the neighbors, whereas numerical study of [33] revealed that in some cases the blow-up might be much weaker, e.g. of $O(|\ln\delta|)$. Such a discrepancy comes from the fact that in the high packing regime the dynamics of particulate flows is driven by the long-range interactions between particles and local asymptotics is not sufficient here, therefore analysis of this concentration regime is quite challenging. Luckily, the development of discrete network approximation techniques of [7–9] allowed to settle the disagreement between the results of [17] and [33]. Namely, it was shown that there are multiple blow-ups of the effective viscosity as $\delta \to 0$ demonstrating examples of the cases when the stronger blow-up degenerates so the weaker blow-up becomes the leading one (see, in particular, [10,9]).

The third concentration regime, when particle distances are of the same order as their sizes, is called the finite or moderate concentration regime. Unlike in the regimes mentioned above, where the extreme properties in some sense facilitated the corresponding analyses (i.e. negligible interactions between particles in the dilute limit and the presence of strong lubrication forces between closely spaced particles that contributed to the blow-up of the effective viscosity in highly concentrated suspensions), the case of finite concentrations is much harder to analyze. Such a regime was treated in [31] where a periodic array of spherical particles was considered. Under the assumption that all inclusions follow the shear motion of the fluid (formula (5) in [31]) it was shown that $\mu^* = O(\epsilon^{-1})$, where ϵ is the distance between periodically distributed particles. This assumption is analogous to the well-known Cauchy–Born hypothesis in solid state physics, which is known to be not always true [18]. Also, a finite concentration regime was considered in [26] where an asymptotic expansion of the effective viscosity was constructed assuming a periodic distribution of particles of