J. Math. Anal. Appl. 420 (2014) 705–736

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A criterion for the explicit reconstruction of a holomorphic function from its restrictions on lines

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ARTICLE INFO

Article history:
Received 7 February 2014
Available online 6 June 2014
Submitted by Richard M. Aron

Keywords:
Complex variables
Approximation theory
Interpolation theory
Divided differences

ABSTRACT

We deal with a problem of the explicit reconstruction of any holomorphic function f on a ball of $\mathbb{C}^2$ from its restrictions on a union of complex lines. The validity of such a reconstruction essentially depends on the mutual repartition of these lines. This criterion can be analytically described and it is also possible to give geometrical sufficient conditions. The motivation of this problem also comes from possible applications in mathematical economics and medical imaging.

Contents

1. Introduction

1.1. Presentation of the problem and first results

1.1.1. General formulation of the problem

In this paper we deal with a problem of the reconstruction of a holomorphic function from its restrictions on analytic submanifolds. f being a holomorphic function on a domain $\Omega \subset \mathbb{C}^n$ and $\{Z_j\}_{j=1}^N$ a family of analytic submanifolds of Ω, we want to find f from the data $f_{|\{Z_j\}_{j=1}^N} := \{f_{|Z_j}\}_{j=1}^N$. One can give interpolating functions $f_N \in \mathcal{O}(\Omega)$ that satisfy $f_{N|\{Z_j\}_{j=1}^N} = f_{|\{Z_j\}_{j=1}^N}$ (for example if Ω is convex, strictly pseudoconvex or $\Omega = \mathbb{C}^n$, see [1]), but generally $f_N \neq f$. Then a natural way is to consider an infinite family of submanifolds $\{Z_j\}_{j=1}^\infty$ and construct the associated interpolating $f_{\{Z_j\}_{j\geq 1}}$ as $\lim_{N\to\infty} f_N$. In this

http://dx.doi.org/10.1016/j.jmaa.2014.06.004
0022-247X/

case the uniqueness of the interpolating function will certainly be guaranteed but without any assurance of the convergence of the sequence $(f_N)_{N\geq 1}$. Moreover, this motivates the research of explicit reconstruction formulas.

1.1.2. An explicit interpolation formula

Here we deal with the case of $\mathbb{C}^2$, $\Omega = B_2(0,r_0) \subset \mathbb{C}^2$ (where $B_2(0,r_0) = \{z \in \mathbb{C}^2,\ |z_1|^2 + |z_2|^2 < r_0^2\}$), and a family of distinct complex lines that cross the origin. Such a family can be described as

$$\big\{\{z \in \mathbb{C}^2,\ z_1 - \eta_j z_2 = 0\}\big\}_{j\geq 1}, \tag{1.1}$$

with $\eta_j \in \mathbb{C}$ all different, that we will simply denote by $\eta = \{\eta_j\}_{j\geq 1}$ (w.l.o.g. we can forget the line $\{z_2 = 0\}$ that is associated with $\eta_0 = \infty$). On the other hand, $f \in \mathcal{O}(B_2(0,1))$ being given, a way to give one interpolating function f_N is the one that uses one of the essential ideas from [1], whose computation exploits residues and principal values (see [2] and [7]) and whose motivation is to get a formula that fixes any polynomial function with degree smaller than N. $S_2(0,1)$ being the unit sphere, one has $\forall z \in B_2(0,1)$,

$$\begin{aligned} f(z) = \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^2} \int\limits_{\zeta\in S_2(0,1),|\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)|=\varepsilon} \frac{f(\zeta)\det(\bar{\zeta}, P_N(\zeta,z))\omega(\zeta)}{\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)(1-\langle\bar{\zeta},z\rangle)} \\ - \lim_{\varepsilon\to 0} \frac{\prod_{j=1}^N(z_1-\eta_j z_2)}{(2\pi i)^2} \int\limits_{\zeta\in S_2(0,1),|\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)|>\varepsilon} \frac{f(\zeta)\omega'(\bar{\zeta})\wedge\omega(\zeta)}{\prod_{j=1}^N(\zeta_1-\eta_j\zeta_2)(1-\langle\bar{\zeta},z\rangle)^2}, \end{aligned}$$

where $\omega'(\zeta) = \zeta_1 d\zeta_2 - \zeta_2 d\zeta_1$, $\omega(\zeta) = d\zeta_1 \wedge d\zeta_2$, and $P_N(\zeta,z) \in (\mathcal{O}(\mathbb{C}^2\times\mathbb{C}^2))^2$ satisfies $\forall(\zeta,z) \in \mathbb{C}^2\times\mathbb{C}^2$,

$$\big\langle P_N(\zeta), \zeta - z\big\rangle = P_{N,1}(\zeta,z)(\zeta_1 - z_1) + P_{N,2}(\zeta,z)(\zeta_2 - z_2) = \prod_{j=1}^N(\zeta_1 - \eta_j\zeta_2) - \prod_{j=1}^N(z_1 - \eta_j z_2).$$

Both integrals can be explicited and yield the following relation: let $f \in \mathcal{O}(B_2(0,r_0))$ (resp. $f \in \mathcal{O}(\mathbb{C}^2)$), one has $\forall z \in B_2(0,r_0)$ (resp. $z \in \mathbb{C}^2$),

$$f(z) = E_N(f;\eta)(z) - R_N(f;\eta)(z) + \sum_{k+l\geq N} a_{k,l} z_1^k z_2^l, \tag{1.2}$$

where $\sum_{k,l\geq 0} a_{k,l} z_1^k z_2^l$ is the Taylor expansion of f,

$$\begin{aligned} E_N(f;\eta)(z) := \sum_{p=1}^N \Bigg(\prod_{j=p+1}^N (z_1 - \eta_j z_2)\Bigg) \sum_{q=p}^N \frac{1+\eta_p\overline{\eta_q}}{1+|\eta_q|^2} \frac{1}{\prod_{j=p,\,j\neq q}^N(\eta_q - \eta_j)} \\ \times \sum_{m\geq N-p} \bigg(\frac{z_2 + \overline{\eta_q}z_1}{1+|\eta_q|^2}\bigg)^{m-N+p} \frac{1}{m!}\frac{\partial^m}{\partial v^m}\bigg|_{v=0} \big[f(\eta_q v, v)\big] \end{aligned} \tag{1.3}$$

and

$$R_N(f;\eta)(z) := \sum_{p=1}^N \Bigg(\prod_{j=1,\,j\neq p}^N \frac{z_1 - \eta_j z_2}{\eta_p - \eta_j}\Bigg) \sum_{k+l\geq N} a_{k,l}\eta_p^k \bigg(\frac{z_2 + \overline{\eta_p}z_1}{1+|\eta_p|^2}\bigg)^{k+l-N+1}. \tag{1.4}$$

This relation is an application of the main theorem from [8] that is a more general version for the case of multiple complex lines (i.e. with the restriction of f and its first derivatives on every line). A direct proof