J. Math. Anal. Appl. 420 (2014) 737–749

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Chebyshev type inequalities for Hilbert space operators

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A R T I C L E I N F O

Article history:
Received 31 July 2013
Available online 29 May 2014
Submitted by D. Blecher

Keywords:
Chebyshev inequality
Hadamard product
Bochner integral
Super-multiplicative function
Singular value
Operator mean

A B S T R A C T

We establish several operator extensions of the Chebyshev inequality. The main version deals with the Hadamard product of Hilbert space operators. More precisely, we prove that if $\mathscr{A}$ is a C^*-algebra, T is a compact Hausdorff space equipped with a Radon measure μ, $\alpha : T \to [0, +\infty)$ is a measurable function and $(A_t)_{t\in T}$, $(B_t)_{t\in T}$ are suitable continuous fields of operators in $\mathscr{A}$ having the synchronous Hadamard property, then

$$\int_T \alpha(s)d\mu(s) \int_T \alpha(t)(A_t \circ B_t)d\mu(t) \geq \left(\int_T \alpha(t)A_t d\mu(t)\right) \circ \left(\int_T \alpha(s)B_s d\mu(s)\right).$$

We apply states on C^*-algebras to obtain some versions related to synchronous functions. We also present some Chebyshev type inequalities involving the singular values of positive $n \times n$ matrices. Several applications are given as well.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ together with the operator norm $\|\cdot\|$. Let I stand for the identity operator. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive (positive semidefinite for a matrix A) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and then we write $A \geq 0$. By a strictly positive operator (positive definite for a matrix) A, denoted by $A > 0$, we mean a positive invertible operator. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $B \geq A$ ($B > A$, resp.) if $B - A \geq 0$ ($B - A > 0$, resp.). For $A \in \mathbb{M}_n$, the singular values of A, denoted by $s_1(A), s_2(A), \cdots, s_n(A)$, are the eigenvalues of the positive matrix $|A| = (A^*A)^{\frac{1}{2}}$ enumerated as $s_1(A) \geq \cdots \geq s_n(A)$ with their multiplicities counted.

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http://dx.doi.org/10.1016/j.jmaa.2014.05.078

The Gelfand map $f(t) \mapsto f(A)$ is an isometric $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by I and A. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$. Let f be a continuous real valued function on an interval J. The function f is called operator monotone (operator decreasing, resp.) if $A \leq B$ implies $f(A) \leq f(B)$ $(f(B) \leq f(A)$, resp.) for all $A, B \in \mathbb{B}_h^J(\mathscr{H})$, where $\mathbb{B}_h^J(\mathscr{H})$ is the set of all self-adjoint operators in $\mathbb{B}(\mathscr{H})$, whose spectra are contained in J; cf. [10].

Given an orthonormal basis $\{e_j\}$ of a Hilbert space $\mathscr{H}$, the Hadamard product $A \circ B$ of two operators $A, B \in \mathbb{B}(\mathscr{H})$ is defined by $\langle A \circ B e_i, e_j\rangle = \langle A e_i, e_j\rangle\langle B e_i, e_j\rangle$. Clearly $A \circ B = B \circ A$. It is known that the Hadamard product can be presented by filtering the tensor product $A \otimes B$ through a positive linear map. In fact,

$$A \circ B = U^*(A \otimes B)U, \tag{1.1}$$

where $U : \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}$ is the isometry defined by $Ue_j = e_j \otimes e_j$; see [6]. It follows from (1.1) that if $A \geq 0$ and $B \geq 0$, then

$$A \circ B \geq 0. \tag{1.2}$$

For matrices, one easily observes [14] that the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is $A \circ B = (a_{ij}b_{ij})$, a principal submatrix of the tensor product $A \otimes B = (a_{ij}B)_{1\leq i,j\leq n}$. From now on when we deal with the Hadamard product of operators, we explicitly assume that an orthonormal basis is fixed.

The axiomatic theory of operator means has been developed by Kubo and Ando [8]. An operator mean is a binary operation σ defined on the set of strictly positive operators, if the following conditions hold:

(i) $A \leq C$, $B \leq D$ imply $A\,\sigma\,B \leq C\,\sigma\,D$;
(ii) $A_n \downarrow A, B_n \downarrow B$ imply $A_n\,\sigma\,B_n \downarrow A\,\sigma\,B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iii) $T^*(A\,\sigma\,B)T \leq (T^*AT)\,\sigma\,(T^*BT)$ $(T \in \mathbb{B}(\mathscr{H}))$;
(iv) $I\,\sigma\,I = I$.

There exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0, \infty)$ with $f(1) = 1$ via $f(t)I = I\,\sigma\,(tI)$ $(t > 0)$. In addition, $A\,\sigma\,B = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$ for all strictly positive operators A, B. The operator monotone function f is called the representing function of σ. Using a limit argument by $A_\varepsilon = A + \varepsilon I$, one can extend the definition of $A\,\sigma\,B$ to positive operators. The operator means corresponding to the operator monotone functions $f_{\sharp_\mu}(t) = t^\mu$ and $f_!(t) = \frac{2t}{1+t}$ on $[0, \infty)$ are the operator weighted geometric mean $A \sharp_\mu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\mu A^{\frac{1}{2}}$ and the operator harmonic mean $A\,!\,B = 2(A^{-1} + B^{-1})^{-1}$, respectively.

Let us consider the real sequences $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n)$ and a non-negative sequence $w = (w_1, \cdots, w_n)$. Then the weighed Chebyshev function is defined by

$$T(w; a, b) := \sum_{j=1}^n w_j \sum_{j=1}^n w_j a_j b_j - \sum_{j=1}^n w_j a_j \sum_{j=1}^n w_j b_j.$$

In 1882, Chebyshev [3] proved that if a and b are monotone in the same sense, then $T(w; a, b) \geq 0$. Some integral generalizations of this inequality were given by Barza, Persson and Soria [1]. The Chebyshev inequality is a complement of the Grüss inequality; see [11] and the references therein.

A related notion is synchronicity. Two continuous functions $f, g : J \to \mathbb{R}$ are called synchronous on an interval J, if