J. Math. Anal. Appl. 420 (2014) 758–771

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Simultaneously continuous retraction and Bishop–Phelps–Bollobás type theorem

Sun Kwang Kim [a], Han Ju Lee [b,*,1]

[a] Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea
[b] Department of Mathematics Education, Dongguk University – Seoul, 100-715 Seoul, Republic of Korea

ARTICLE INFO

Article history:
Received 29 January 2014
Available online 11 June 2014
Submitted by B. Cascales

Keywords:
Banach space
Approximation
Retraction
Norm-attaining operators
Bishop–Phelps–Bollobás theorem

ABSTRACT

The dual space X^* of a Banach space X is said to admit a uniformly simultaneously continuous retraction if there is a retraction r from X^* onto its unit ball B_{X^*} which is uniformly continuous in norm topology and continuous in weak-$*$ topology. We prove that if a Banach space (resp. complex Banach space) X has a normalized unconditional Schauder basis with unconditional basis constant 1 and if X^* is uniformly monotone (resp. uniformly complex convex), then X^* admits a uniformly simultaneously continuous retraction. It is also shown that X^* admits such a retraction if $X = [\bigoplus X_i]_{c_0}$ or $X = [\bigoplus X_i]_{\ell_1}$, where $\{X_i\}$ is a family of separable Banach spaces whose duals are uniformly convex with moduli of convexity $\delta_i(\varepsilon)$ with $\inf_i \delta_i(\varepsilon) > 0$ for all $0 < \varepsilon < 1$. Let K be a locally compact Hausdorff space and let $C_0(K)$ be the real Banach space consisting of all real-valued continuous functions vanishing at infinity. As an application of simultaneously continuous retractions, we show that a pair $(X, C_0(K))$ has the Bishop–Phelps–Bollobás property for operators if X^* admits a uniformly simultaneously continuous retraction. As a corollary, $(C_0(S), C_0(K))$ has the Bishop–Phelps–Bollobás property for operators for every locally compact metric space S.

1. Introduction

Let X be a real or complex Banach space and A be a subset of X. A continuous function $r : X \to A$ is said to be a *retraction* if r is the identity on A. Retractions have various applications in nonlinear geometric functional analysis [11,10,12]. Benyamini introduced the notion of simultaneously continuous retraction from the dual space X^* onto B_{X^*}. More precisely, the dual space X^* of a Banach space X is said to *admit a (resp. uniformly) simultaneously continuous retraction* if there is a retraction r from X^* onto B_{X^*} which is both weak-$*$ continuous and norm continuous (resp. uniformly norm-continuous). Benyamini [11] showed,

* Corresponding author.
E-mail addresses: sunkwang@kgu.ac.kr (S.K. Kim), hanjulee@dongguk.edu (H.J. Lee).

1 The author partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A1006869).

http://dx.doi.org/10.1016/j.jmaa.2014.06.009
0022-247X/

in particular, that E^* admits uniformly simultaneously continuous retraction if E^* is a separable uniformly convex space, or E is the space $C(K)$ of all real-valued continuous functions on a compact metric space K.

As remarked in Proposition 4.22. [12], there is a connection between simultaneously continuous retractions and the denseness of norm attaining operators into $C(K)$. In this paper, we deal with the existence of uniformly simultaneous continuous retraction in a certain Banach space and its applications to Bishop–Phelps–Bollobás type theorem.

2. Uniformly simultaneously continuous retraction

Let $\{e_j\}$ be a normalized unconditional Schauder basis for X with unconditional basis constant 1. Its biorthogonal functionals will be denoted by $\{e_j^*\}$. In fact, it is easy to see that X and X^* are Banach lattices and, for every $x^* \in X^*$, we have

$$x^* = \text{weak} * \sum_{j=1}^{\infty} x^*(j)e_j^*,$$

where $x^*(j) = \langle x^*, e_j \rangle$. Recall that a Banach lattice X is uniformly monotone if, for all $\varepsilon > 0$,

$$M(\varepsilon) = \inf\bigl\{\bigl\||x| + |y|\bigr\| - 1 : \|x\| = 1, \|y\| \geq \varepsilon\bigr\} > 0.$$

It is easy to check that $\varepsilon \mapsto M(\varepsilon)$ is a monotone increasing function and $M(\varepsilon) \leq \varepsilon$ for all $\varepsilon > 0$. This M is called the *modulus of monotonicity* of X. It is easy to check that if X is uniformly monotone, then X is strictly monotone. That is, $\||x| + |y|\| > \|x\|$ for all $x \in X$ and for all nonzero element y in X. The uniform monotonicity of a Banach lattice is equivalent to the uniform complex convexity of its complexification [30,31]. The complex convexity has been used to study density of norm-attaining operators between Banach spaces [1,18].

Benyamini showed [11] that if X has a shrinking Schauder basis $\{e_j\}$ with $\{e_j^*\}$ being strictly monotone, then X^* admits a simultaneously continuous retraction. It is also shown that for $X = \ell_p$, $1 \leq p < \infty$ or $X = c_0$, X^* admits a uniformly simultaneously continuous retraction.

For $t \geq 0$, we define $M^{-1}(t) = \sup\{\varepsilon \geq 0 : M(\varepsilon) \leq t\}$ for a monotone increasing function M. The modulus of continuity for a function φ is defined by

$$\omega_\varphi(t) = \sup\bigl\{\bigl\|\varphi(x^*) - \varphi(y^*)\bigr\| : \bigl\|x^* - y^*\bigr\| \leq t\bigr\}.$$

Let f be a nonnegative function on a deleted neighborhood of 0 with $\lim_{t\to 0+} f(t) = 0$. We say that X^* admits *an f-uniformly simultaneously continuous retraction* if there is a uniformly simultaneously continuous retraction φ with $\omega_\varphi(t) \leq f(t)$.

Theorem 2.1. *Suppose that a Banach space X has a normalized unconditional Schauder basis $\{e_j\}$ with unconditional basis constant 1. If X^* is uniformly monotone with modulus of monotonicity M, then X^* admits a uniformly simultaneously continuous retraction with modulus of continuity $2M^{-1}$.*

Proof. Notice that X^* is uniformly monotone and it is order-continuous (cf. [30]) and $\{e_j^*\}_{j=1}^\infty$ is a Schauder basis. Given $x^* = \sum_{j=1}^\infty a_j e_j^*$ with $x^* \notin B_{X^*}$, there is a unique n so that

$$\Biggl\|\sum_{j=1}^{n-1} a_j e_j^*\Biggr\| < 1, \quad \text{and} \quad \Biggl\|\sum_{j=1}^{n} a_j e_j^*\Biggr\| \geq 1.$$

By the strict monotonicity and convexity of norm, there is a unique $0 < t \leq 1$ so that