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Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions

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ABSTRACT

In this note we present some uniqueness and comparison results for a class of problem of the form

$$-Lu = H(x, u, \nabla u) + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain, L is a general elliptic second order linear operator with bounded coefficients and H is allowed to have a critical growth in the gradient. In some cases our assumptions prove to be sharp.

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ and a function $h \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we consider the problem

$$-Lu = H(x, u, \nabla u) + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{1.1}$$

where L is a general elliptic second order linear operator and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfy the assumptions:

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(L) There exists a family of functions $(a^{ij})_{1\le i,j\le N}$ with $a^{ij} \in L^\infty(\Omega) \cap W^{1,\infty}_{loc}(\Omega)$ such that

$$Lu = \sum_{i,j} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right)$$

and, there exists $\eta > 0$ such that, for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$,

$$\sum_{i,j} a^{i,j}(x)\xi_i\xi_j \ge \eta|\xi|^2.$$

(H1) There exists a continuous function $C_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $b_1 \in L^p(\Omega)$ such that, for a.e. $x \in \Omega$, all $u \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$,

$$\left|H(x,u,\xi)\right| \le C_1\big(|u|\big)\big(|\xi|^2 + b_1(x)\big).$$

(H2) There exists a function $b_2 \in L^N_{loc}(\Omega)$ and a continuous function $C_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ such that, for a.e. $x \in \Omega$, all $u_1, u_2 \in \mathbb{R}$ with $u_1 \ge u_2$ and all $\xi_1, \xi_2 \in \mathbb{R}^N$,

$$H(x,u_1,\xi_1) - H(x,u_2,\xi_2) \le C_2\big(|u_1|,|u_2|\big)\big(|\xi_1| + |\xi_2| + b_2(x)\big)|\xi_1 - \xi_2|.$$

As we shall see in the proof of Corollary 2.1, a sufficient condition for (H2) is that for a.e. $x \in \Omega$, $H(x,\cdot,\cdot) \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^N)$ with

$$\frac{\partial H}{\partial u}(x,u,\xi) \le 0, \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{1.2}$$

and that there exists a function $b_3 \in L^N_{loc}(\Omega)$ and a continuous nondecreasing function $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\left|\frac{\partial H}{\partial \xi}(x,u,\xi)\right| \le C\big(|u|\big)\big(|\xi| + b_3(x)\big), \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N. \tag{1.3}$$

Uniqueness of solution for problem (1.1) (with $Lu = \Delta u$) has been first studied in the work [4] and after improved in [3] by requiring weaker regularity conditions on the data. The reader can also see additional uniqueness results in [5] for subcritical nonlinear term H (with respect to ξ), i.e., when its growth is less than a power $|\xi|^q$ with $q < 2$, and in the work [2] for the case that H has a singularity at $u = 0$.

Specifically, in [3] the uniqueness of solution for every h is proved when it is assumed condition (1.3) and the following strengthening of (1.2):

$$\frac{\partial H}{\partial u}(x,u,\xi) \le -d_0 < 0, \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

However, in the case that it is only assumed the general hypothesis (1.2) (together with (1.3)), the authors require to the function h to be sufficiently small in an appropriate sense. Furthermore, adapting the arguments of [3], the case where (1.2)–(1.3) hold and h has sign can also be covered. Nevertheless, the treatment of the general case (1.2)–(1.3) with no assumptions on h seems out of reach with the approach of [3,4].

The special case of (1.1) given by

$$-\Delta u = d(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H^1_0(\Omega) \cap L^\infty(\Omega) \tag{1.4}$$

is studied in [1] by an alternative approach. Indeed, if $d, h \in L^p(\Omega)$ for some $p > \frac{N}{2}$, $\mu \in L^\infty(\Omega)$, then it is proved that (1.4) has at most one solution as soon as $d \le 0$. Actually this condition is also necessary since