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Self-adjointness of unbounded tridiagonal operators and spectra of their finite truncations

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ABSTRACT

This paper addresses two different but related questions regarding an unbounded symmetric tridiagonal operator: its self-adjointness and the approximation of its spectrum by the eigenvalues of its finite truncations. The sufficient conditions given in both cases improve and generalize previously known results. It turns out that, not only self-adjointness helps to study limit points of eigenvalues of truncated operators, but the analysis of such limit points is a key help to prove self-adjointness. Several examples show the advantages of these new results compared with previous ones. Besides, an application to the theory of continued fractions is pointed out.

1. Introduction

Symmetric tridiagonal matrices provide the canonical matrix representations of self-adjoint operators in Hilbert spaces [25] and, as a consequence, they naturally emerge in phenomena governed by self-adjoint operators. On the other hand, self-adjoint operators are ubiquitous in practical applications because of the usual requirement of a real spectrum in physical problems. Due to these reasons, symmetric tridiagonal operators appear in many areas of mathematics and physics.

A symmetric tridiagonal operator T in an infinite dimensional Hilbert space $(H,(\cdot,\cdot))$ with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ is given without loss by

$$Te_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1}, \quad a_n > 0,\ b_n \in \mathbb{R},\ n = 1, 2, \ldots, \tag{1.1}$$

where $e_0 = 0$. The matrix representation of T in the basis $\{e_n\}_{n=1}^{\infty}$ is

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$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & b_4 & a_4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{1.2}$$

which is known as a Jacobi matrix. It is assumed that $a_n > 0$ because the complex conjugated upper and lower diagonals can be made non-negative by a change of basis $e_n \to \eta_n e_n$, $|\eta_n| = 1$, while setting $a_n = 0$ for some n splits (1.2) into a direct sum of Jacobi matrices that can be analyzed independently.

If P_N is the orthogonal projection onto the subspace $H_N = \operatorname{span}\{e_n\}_{n=1}^N$, the composition $T_N = P_N T P_N$ defines an operator in H_N called the orthogonal truncation of T on H_N. Its matrix representation in the basis $\{e_n\}_{n=1}^N$ is the principal submatrix of (1.2) of order N.

Expression (1.1) defines a symmetric operator in the linear span of $\{e_n\}_{n=1}^\infty$, but we will identify T with the closure of such an operator, which is known to be symmetric too. Then, either T is self-adjoint, or T has infinitely many self-adjoint extensions. In the latter case, the self-adjoint extensions have pure point spectra with any two disjoint [24, Theorem 4.11]. Different self-adjoint extensions can appear only when T is unbounded, which is equivalent to saying that some of the sequences a_n or b_n is unbounded. Thus, self-adjointness is non-trivial only in the unbounded case, which is also of practical interest since unbounded operators naturally appear in applications.

The general spectral problem for unbounded Jacobi matrices and, more specifically, approximation problems concerning such a spectrum have been considered in several studies. Indicatively we mention the recent works [9,11,14–23,26].

The present paper deals with two closely related problems concerning unbounded symmetric tridiagonal operators T: the search for self-adjointness conditions for T which go further than known ones, and the possibility of approximating the spectrum $\sigma(T)$ of T via the spectra $\sigma(T_N)$ of its orthogonal truncations T_N. To be more precise, let us denote by $\Lambda(T)$ the set of all limit points of the eigenvalues of T_N when $N \to \infty$, i.e.

$$\Lambda(T) = \bigl\{\lambda \in \operatorname*{Lim}_{N\to\infty} \lambda_N : \lambda_N \in \sigma(T_N)\bigr\},$$
$$\operatorname*{Lim}_{N\to\infty} \lambda_N = \text{set of limit points of the sequence } \lambda_N.$$

Information about $\Lambda(T)$ is of great importance not only from the point of view of operator theory, but also for the theory of continued fractions, orthogonal polynomials and numerical analysis (see [12] and the references therein).

In particular, the eigenvalues of T_N coincide with the zeros of the polynomial $p_{N+1}(x)$ given by the recurrence relation

$$a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) = x p_n(x), \quad n = 1, 2, \dots, \tag{1.3}$$

with $p_0(x) = 0$ and $p_1(x) = 1$. Thus, $\Lambda(T)$ coincides with the set of limit points of the zeros of the orthogonal polynomials $p_n(x)$ satisfying (1.3).

Besides, if T is self-adjoint, the Jacobi continued fraction

$$K(\lambda) = \frac{1}{|\lambda - b_1} - \frac{a_1^2}{|\lambda - b_2} - \frac{a_2^2}{|\lambda - b_3} - \cdots \tag{1.4}$$

converges to the function $((\lambda - T)^{-1} e_1, e_1)$ for every $\lambda \in \mathbb{C} \setminus \Lambda(T)$ [5,12].

The self-adjointness of T ensures the inclusion $\Lambda(T) \supseteq \sigma(T)$, although in general it does not guarantee the equality $\Lambda(T) = \sigma(T)$ (see for instance [1,5,12,13,25], and also [6, Proposition 2.1] for a generalization to